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Monte Carlo Studies of Non-Abelian Gauge Theories†

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Abstract

After some general remarks on the efficiency of various Monte Carlo algorithms for gauge theories, I discuss the calculation of the asymptotic freedom scales of SU(2) and SU(3) gauge theories in the absence of quarks. There are large numerical factors between these scales when defined in terms of the bare coupling of the lattice theory or when defined in terms of the physical force between external sources.

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Abstract

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Recently Monte Carlo techniques have proven to be a powerful non-perturbative tool in the study of quantized gauge theories.[1,2,3] In this talk I will first make some general remarks on Monte Carlo algorithms and then discuss calculation of the asymptotic freedom scales in SU(2) and SU(3) quarkless Yang-Mills theories.

The basic goal of a Monte Carlo procedure for a system of degrees of freedom ϕ_i is to stochastically generate a sequence of configurations C_j such that asymptotically any configuration C has probability density

$$P(C) \sim e^{-\beta S(C)} \tag{1}$$

Here S(C) is the action and β the inverse coupling constant. Expectation values of products of the ϕ_i are identified with the Green's functions of the theory defined by a path integral. In statistical mechanics, S becomes the Hamiltonian and β the inverse temperature.

In practice, useful Monte Carlo algorithms are based on a principle of detailed balance. Let $P(C \to C')$ denote the probability that a given configuration C in the sequence yields C' as the next configuration. Suppose P satisfies

$$P(C \to C')e^{-\beta S(C)} = P(C' \to C)e^{-\beta S(C')}. \tag{2}$$

It is straightforward to define a norm between ensembles of configurations such that application of the Monte Carlo procedure determined by P will reduce the distance between any given ensemble and the equilibrium ensemble satisfying Eq. (1).[4] If the algorithm has eventual access to any configuration, then ultimately Eq. (1) will be approached.

Eq. (2) is a rather general condition which leaves open considerable leeway in prescription. For simplicity, usually only one degree of freedom ϕ_i

is varied at a time. In this case, the most intuitive algorithm is to replace the given ϕ_i with a new value chosen randomly from all allowed ϕ_i with weight proportional to the exponentiated action in Eq. (1). Physically, this corresponds to placing a heat bath at inverse temperature β in contact with the variable in question. This process is repeated on all the variables of the system and then the entire procedure is iterated.

In processing a given field variable, this heat bath method reduces the above mentioned norm of the distance of an ensemble from equilibrium by the greatest amount in comparison with all other algorithms working on the same variable. This follows because repeated application of any algorithm to a single variable must eventually approach the heat bath. Unfortunately, the generation of the appropriately weighted variable may be rather complicated and time consuming. Consequently, considerably computationally simpler algorithms satisfying Eq. (2) have been devised.[4] Nonetheless, for continuous variables lying in simple manifolds, such as U(1) and SU(2), implementation of a heat bath algorithm can result in considerable savings of computer time.[2] This is particularly the case in gauge theories, where merely combining the interacting neighbor variables is a major part of the computation. Except for the SU(3) calculations, my own Monte Carlo work on lattice gauge theory has been entirely with the heat bath technique. With SU(3) I have used a variation of the Metropolis et al.[5] procedure.

I now turn to a discussion of the scales of asymptotic freedom. In a non-Abelian gauge theory the bare charge goes to zero with the logarithm of the cutoff[6]

$$g_0^{\,2} = \frac{1}{\gamma_0 \log\left(\frac{1}{\Lambda_0^2 a^2}\right) + \frac{\gamma_1}{\gamma_0} \log\left(\log\left(\frac{1}{\Lambda_0^2 a^2}\right)\right) + O\left(\frac{1}{\log\left(\frac{1}{\Lambda_0^2 a^2}\right)}\right)} \qquad (3)$$

Here g_0 is the bare charge, γ_0 and γ_1 are the first two coefficients in the perturbative expansion of the Gell-Mann Low function

$$a\frac{dg_0(a)}{da} = \gamma(g_0) = \gamma_0 g_0^3 + \gamma_1 g_0^5 + O(g^7) \qquad (4)$$

and a is a minimum length imposed as an ultraviolet cutoff. In the lattice formulation a is the lattice spacing. The parameter Λ_0 is the asymptotic freedom scale associated with whatever renormalization scheme is being used. For SU(N) gauge groups the coefficients in Eq. (4) are

$$\gamma_0 = \frac{11}{3}\frac{N}{16\pi^2} \qquad (5)$$

$$\gamma_1 = \frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2 \qquad (6)$$

These first two coefficients are independent of renormalization prescription. The scale Λ_0 is defined by Eq. (3), which can be rewritten

$$\Lambda_0 = \lim_{a\to 0}\frac{1}{a}\left(\gamma_0 g_0^2(a)\right)^{-\frac{\gamma_1}{2\gamma_0^2}} \exp\left(-\frac{1}{2\gamma_0 g_0^2(a)}\right) \qquad (7)$$

My Monte Carlo estimates of these scales for the pure gauge theories are

$$\Lambda_0 = (1.3\pm.2)\times 10^{-2}\sqrt{K} \qquad SU(2) \qquad (8)$$

$$\Lambda_0 = (5.0\pm1.5)\times 10^{-3}\sqrt{K} \qquad SU(3) \qquad (9)$$

Here, Wilson's lattice regularization is used and K is the string tension, the coefficient of the linear potential between widely separated sources in the fundamental representation of the gauge group. I will return later to the calculation of these numbers.

At first sight, these small numbers are rather surprising, coming as they do from a theory with no free parameters. However the value of Λ_0 is not independent of renormalization scheme. Since it is defined in a weak coupling limit, one loop perturbative calculations can relate different definitions of Λ_0. Hasenfratz and Hasenfratz[8] have recently done a lengthy calculation to relate this lattice Λ_0 to the more conventional scale Λ^{MOM} defined by a momentum space subtraction procedure in the Feynman gauge. Their results are

$$\Lambda^{MOM} = 57.5 \quad \Lambda_0 \qquad SU(2) \tag{10}$$

$$\Lambda^{MOM} = 83.5 \quad \Lambda_0 \qquad SU(3) \tag{11}$$

These large coefficients partially cancel the small numbers in Eqs. (8-9). If we accept the string model connection with the Regge slope[9]

$$K = \frac{1}{2\pi\alpha'} \tag{12}$$

and use $\alpha' = 1.0\ (\mathrm{GeV})^{-2}$, then we conclude for SU(3)

$$\Lambda^{MOM} = 170\pm 50\ \mathrm{MeV} \tag{13}$$

Some caution may be necessary in the phenomenological interpretation of this number because I have not included any effects of light quarks.

For the gauge group SU(2) I have performed a Monte Carlo check on the large factor occurring in Eq. (10). I consider a physical renormalization scale defined in terms of the force between quarks. Rectangular Wilson loops with one long dimension measure the energy of a quark-antiquark pair separated by the shorter dimension

$$W(I,J) \to e^{-E(Ia)Ja} \qquad J \gg I \gg 1 \tag{14}$$

Here $W(I,J)$ is a rectangular Wilson loop of size I by J in lattice units and $E(Ia)$ is the energy of two sources in the fundamental representation of the gauge group and separated by a distance Ia. From this I obtain

$$E(aI) = -\frac{1}{a}\log\frac{W(I,J)}{W(I,J-1)}, \qquad J \gg I \gg 1 \tag{15}$$

Unfortunately this energy includes the self energy of the sources and therefore will diverge as the cutoff is removed.[10] To remove this divergence, work with the force

$$F\left(a\sqrt{I(I-1)}\right) = \frac{1}{a}\Big(E(aI) - E(a(I-1))\Big)$$

$$= -\frac{1}{a^2}\log\frac{W(I,J)\,W(I-1,J-1)}{W(I,J-1)\,W(I-1,J)}$$

$$\equiv \frac{1}{a^2}\chi(I,J) \tag{16}$$

where this defines the function $\chi(I,J)$. I have arbitrarily selected the geometric mean of I and (I-1) to define the physical distance. In the limit of short distances, a simple perturbative calculation gives

$$F(r) = \frac{3g_R^2(r)}{16\pi r^2} \tag{17}$$

for SU(2). Here $g_R^2(r)$ is the charge renormalized at scale r. Indeed, the force represents a natural gauge invariant <u>definition</u> of a renormalized charge. Thus I define for $J \gg I \gg 1$

$$g_R^2\,(\sqrt{I(I-1)}\;a) = \frac{16\pi}{3}\,I(I-1)\chi(I,J) \tag{18}$$

In a Monte Carlo calculation, practical lattice sizes do not allow $J \gg I \gg 1$. Indeed, I am forced to study Eq. (18) with I of only 2 or 3 and J up to 5. Asymptotic freedom, in relating g^2 on different scales, allows a check on this optimistic extension of Eq. (18). The renormalization group predicts

$$\beta_R(\sqrt{6}\;a) = \beta_R(\sqrt{2}\;a) - \frac{11}{6\pi^2}\log 3 + O\left(\frac{1}{\beta_R}\right) \tag{19}$$

where I have defined the inverse renormalized coupling

$$\beta_R(r) = \frac{4}{g_R^2(r)}\;. \tag{20}$$

The factor of 4 is inserted to correspond to the normalization used in Ref. (2) of the inverse coupling

$$\beta = \frac{4}{g_0^2} \tag{21}$$

which represents the inverse temperature of the equivalent statistical system.

In Figure (1) I plot Monte Carlo measurements on a 10^4 lattice of $\beta_R(\sqrt{2}a)$ versus $\beta_R(\sqrt{6}a)$. Here the parameter J is 5 for the largest β and is gradually reduced at stronger coupling where large loops have large statistical errors. The plotted error bars are the standard deviation of the mean over an ensemble of five configurations after attaining equilibrium. In the figure I also plot the predicted asymptotic shift of Eq. (19) and the strong coupling limit

$$\beta_R(\sqrt{6}\;a) = \tfrac{1}{3}\beta_R(\sqrt{2}\;a) + O(\beta^3) \tag{22}$$

For $\beta_R > 1.5$ the figure shows a linear relation between $\beta_R(\sqrt{2}\,a)$ and $\beta_R(\sqrt{6}\,a)$. The asymptotic freedom prediction of the intercept has the correct sign, but is off slightly in magnitude. We interpret this as meaning that using Eq. (18) for I of only 2 induces an error in β_R or order 0.2.

Note that Fig. (1) gives no indication of an equality of $\beta_R(\sqrt{6}\,a)$ with $\beta_R(\sqrt{2}\,a)$ at any finite, non-zero value of coupling. As we are comparing a physical quantity on two different length scales, such an equality would be evidence for a renormalization group fixed point. The absence of such a point is strong additional support for the absence of a phase transition in SU(2) lattice gauge theory. Wilson's more detailed real space renormalization group calculations should strengthen this conclusion.[3]

In Fig. (2), we plot $\beta_R(\sqrt{2}\,a)$ as measured above versus the inverse bare charge β. For β larger than 2.5 a good description is

$$\beta_R(\sqrt{2}\,a) = \beta(a) - 1.6. \tag{23}$$

The renormalization scale of g_0 was defined in Eq. (7). A renormalized asymptotic freedom scale Λ_R follows from a similar equation in terms of g_R

$$\Lambda_R = \lim_{r\to 0}\left\{ \frac{1}{r}\left(\gamma_0 g_R^{\,2}(r)\right)^{-\frac{\gamma_1}{2\gamma_0^{\,2}}} \exp\left(-\frac{1}{2\gamma_0 g_R^{\,2}(r)}\right)\right\} \tag{24}$$

The difference between Λ_R and Λ_0 is directly measured in Eq. (23)

$$\beta_R(\sqrt{2}\,a) = \beta(a) + \frac{11}{3\pi^2}\log\left(\frac{\Lambda_0}{\sqrt{2}\,\Lambda_R}\right) + O(\beta^{-1}) \tag{25}$$

Thus, I conclude

$$\Lambda_R = \frac{\Lambda_0}{\sqrt{2}}\exp\left\{(1.6\pm.2)\frac{3\pi^2}{11}\right\} = (52\pm28)\Lambda_0 \tag{26}$$

The error is a subjective estimate. Note the remarkable agreement with the Hasenfratz result in Eq. (10).

Finally I discuss how I obtained Λ_0 as quoted in Eqs. 8 and 9. If I and J are large enough that the area law dominates Wilson loops, then for fixed $g_0^{\,2}(a)$

$$\chi(I,J) \rightarrow a^2K \tag{27}$$

where K is the string tension defined below Eq. (9). In the strong coupling regime, $\beta < 2$, $\chi(I,J)$ is essentially independent of I and J and measures the string tension. However as β is increased smaller values of I and J yield a value of χ which begins to deviate from a^2K and go over into a perturbative inverse β behavior.

Thus I expect the true value of a^2K to be given by the envelope of the curves of $\chi(I,J)$ as functions of $\frac{1}{g_0^2}$. In figures (3) and (4) I show the results of Monte Carlo measurements of $\chi(I,I)$ for SU(2) and SU(3) lattice gauge theory. For SU(2) I have used the heat bath method on a 10^4 lattice except in the strong coupling regions where an 8^4 lattice suffices. For SU(3) I use a Metropolis type algorithm on a 4^4 lattice except at g_0^{-2}=1.11 and 1.80 where a 6^4 lattice was used. On these graphs I include curves of the strong coupling limit

$$X(I,J) \xrightarrow[\beta\to 0]{} \begin{cases} \log(3g_0^2) & SU(3) \\ \log(g_0^2) & SU(2) \end{cases} \tag{28}$$

In addition I plot bands corresponding to the asymptotic freedom prediction

$$a^2K \underset{a\to 0}{\to} \frac{K}{\Lambda_0^2}(\gamma_0 g_0^2)^{-\frac{\gamma_1}{\gamma_0^2}} \exp\left\{-\frac{1}{\gamma_0 g_0^2}\right\} \tag{29}$$

with the Λ_0 values quoted in Eqs. (8) and (9). The errors are subjective estimates. The conclusions for SU(3), being based on a rather small lattice, assume a similar structure to that seen with SU(2). Note that the onset of the strong coupling behavior sets in rather abruptly for SU(3) at $g^2 \sim 1$. This agrees well with the series results in Ref. (11).

The results in Figures (2) and (3) can be combined to give the quark-antiquark force in SU(2) gauge theory as a function of separation. Fig. (5) shows the dimensionless ratio $F(r)/K$ as a function of $r\sqrt{K}$. The force is determined from Eq. (17) using the Monte Carlo results for $g_R^2(\sqrt{2}\,a)$. For $r\sqrt{K} < .4$ the radius is taken from Eq. (29) while for larger r I use Eq. (27) with the largest loops giving manageable fluctuations. For large r the figure shows the constant force corresponding to a linear potential while at small r the inverse square law appears with logarithmic corrections. The empirical curve $F/K = 1.0 + .12(r^2K)$ is included in the figures to show that this simple form, which was advocated in Ref. (12), adequately parametrizes the force for a wide range of r. The errors in this figure are statistical only.

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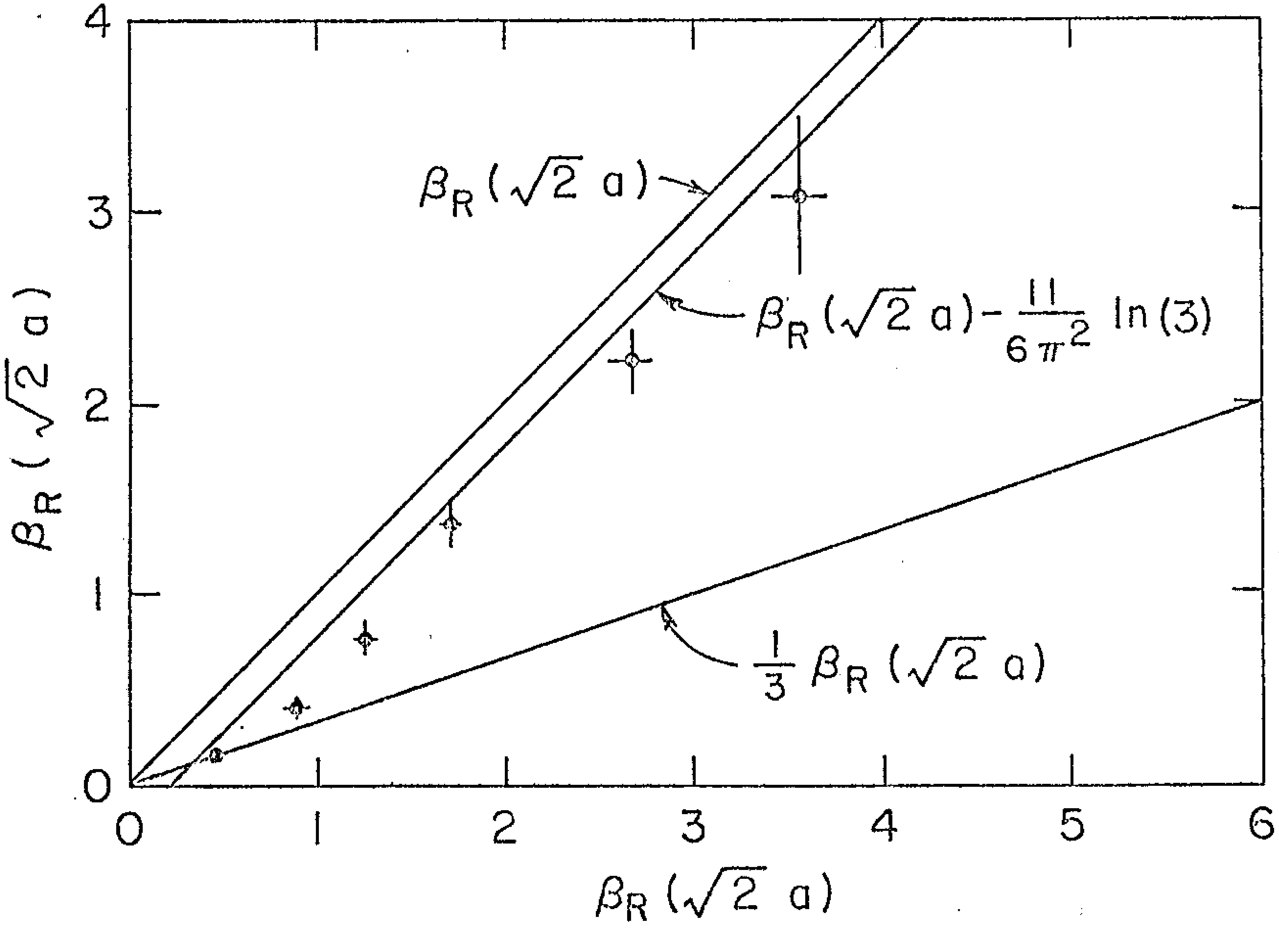
Figure Captions

1. Comparing the renormalized inverse charge at two length scales.
2. The coupling from the potential versus the bare lattice coupling.
3. The quantities X(I,I) for SU(2) gauge theory as a function of $1/g_0^2$. The envelope of these curves describes the string tension as a function of bare coupling.
4. The quantities X(I,I) for SU(3) gauge theory.
5. The interquark force in SU(2) gauge theory.



$\beta_R(\sqrt{2}\,a)$
$\beta_R(\sqrt{2}\,a) - \frac{11}{6\pi^2}\ln(3)$
$\frac{1}{3}\beta_R(\sqrt{2}\,a)$
$\beta_R(\sqrt{2}\,a)$
$\beta_R(\sqrt{2}\,a)$
0
1
2
3
4
5
6
0
1
2
3
4

Fig. 1

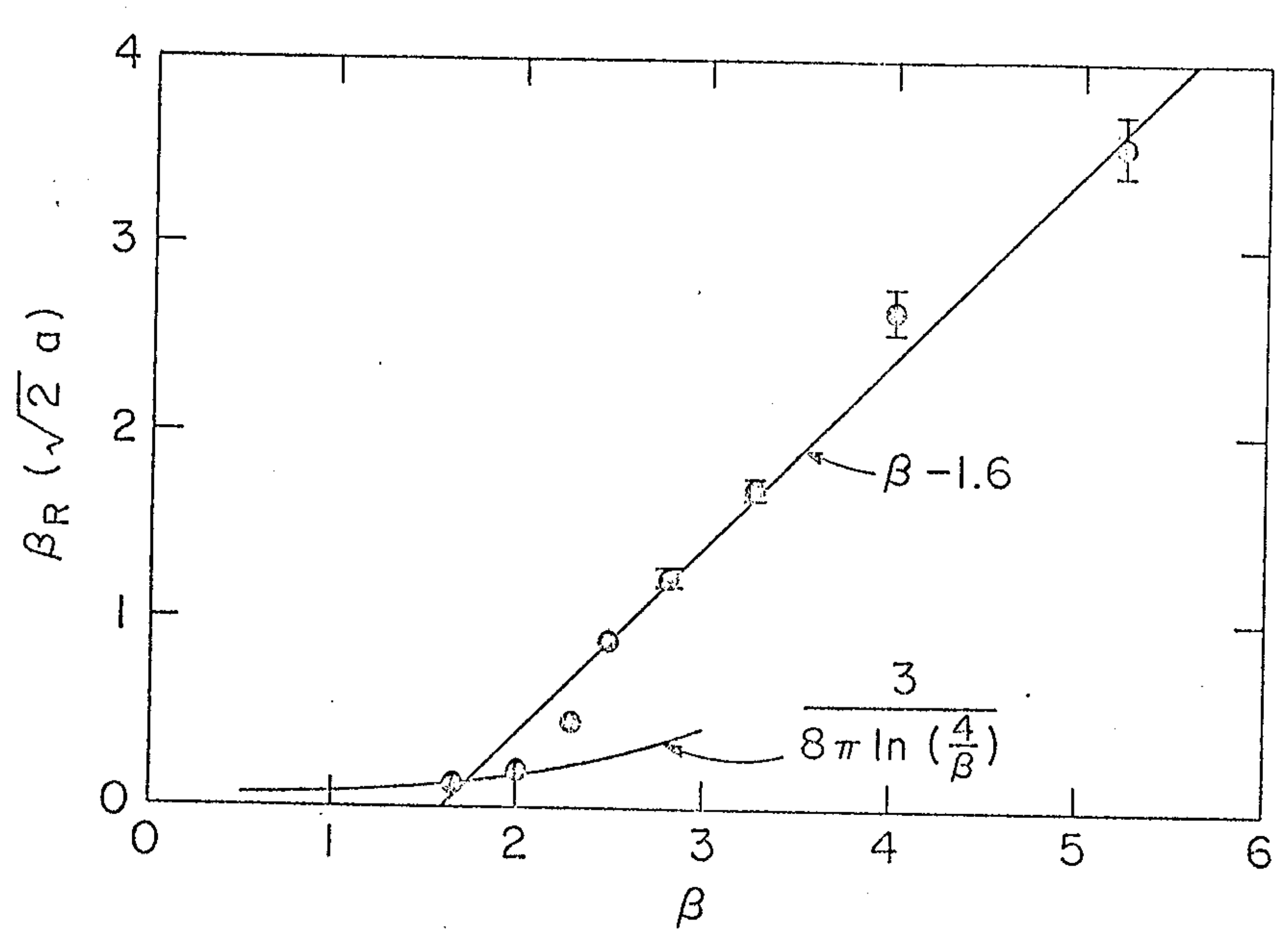
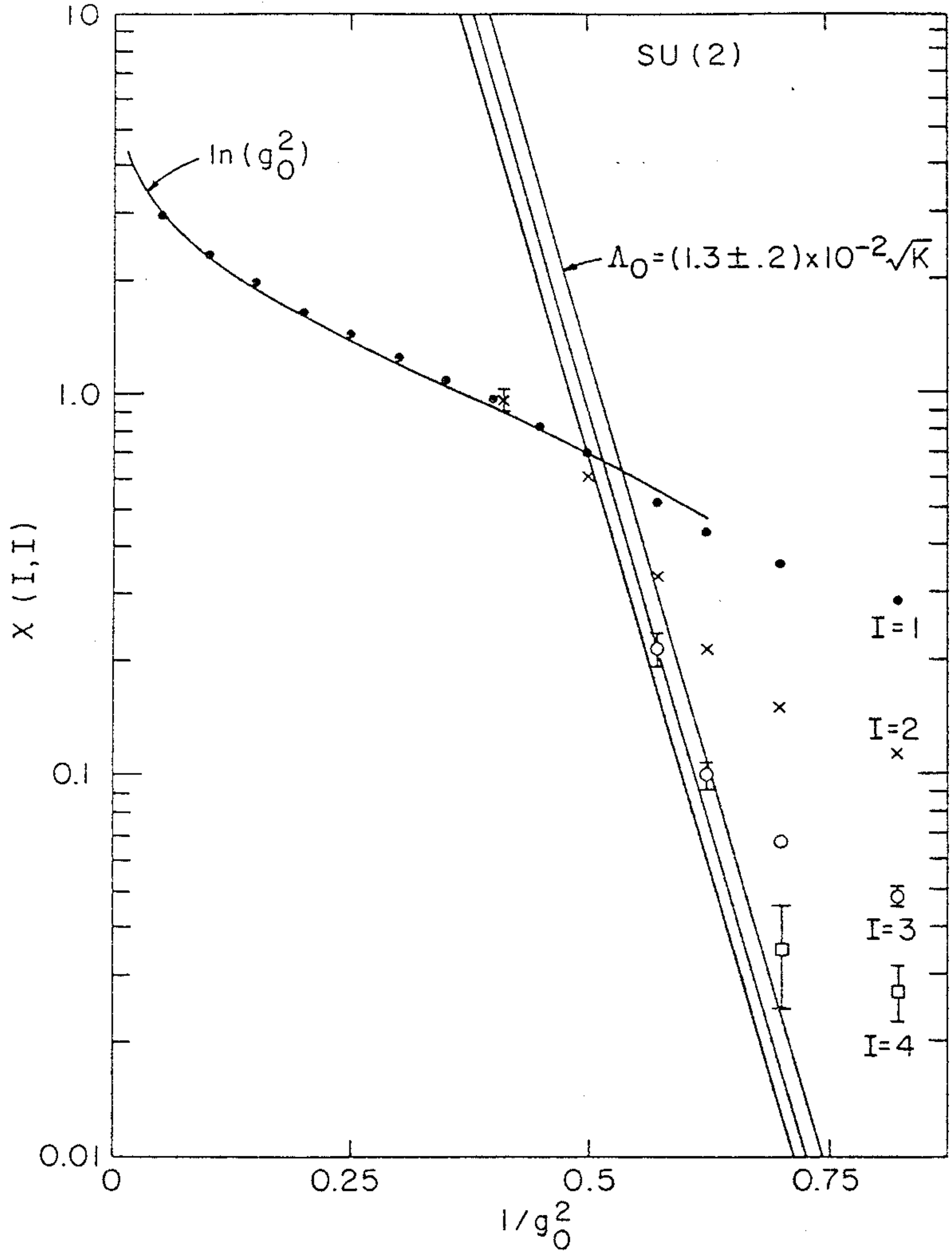
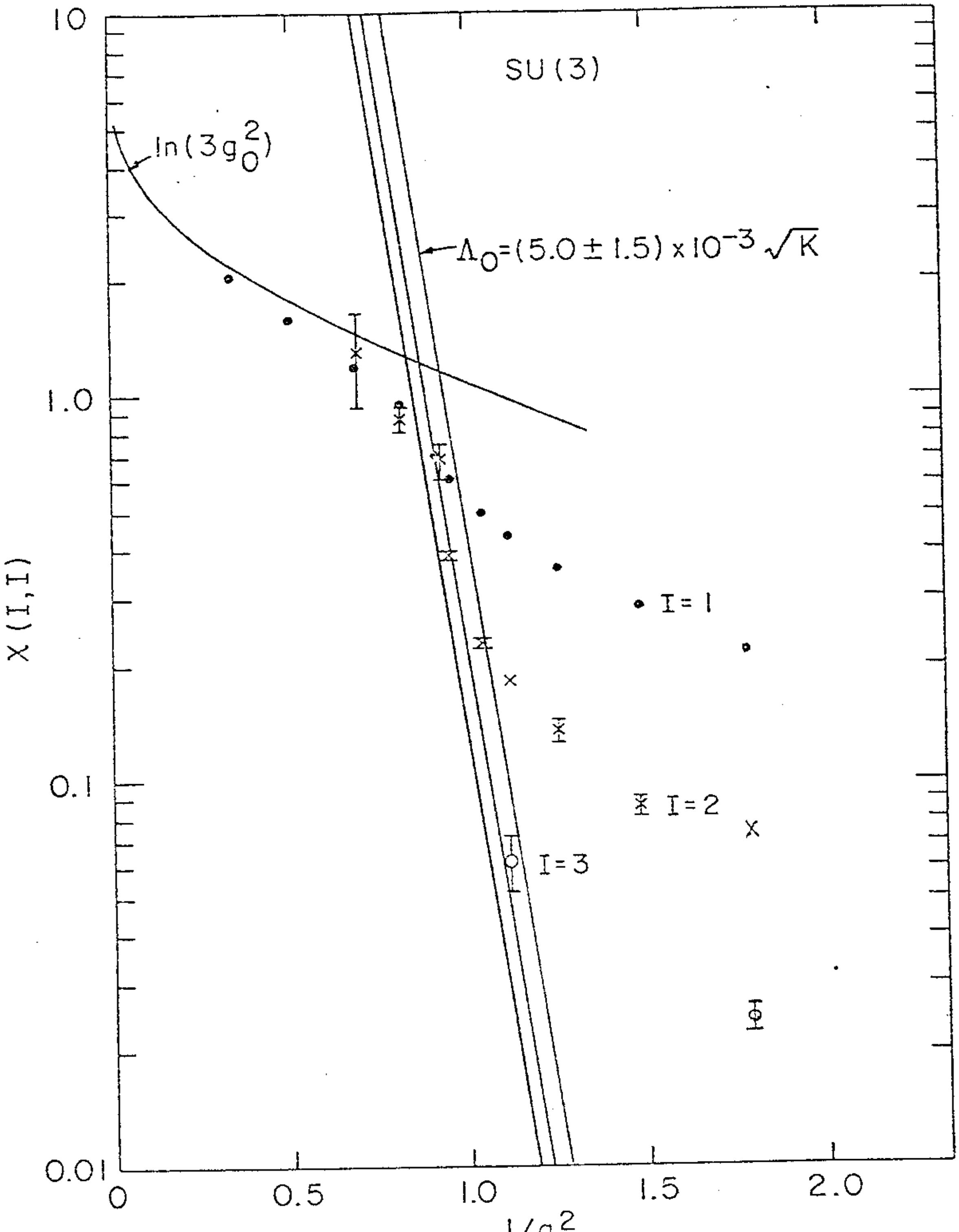


Fig. 2



SU(2)
ln(g_0^2)
Λ_0=(1.3±.2)x$10^{-2}\sqrt{K}$
X (I,I)
$1/g_0^2$
I=1
I=2
I=3
I=4
10
1.0
0.1
0.01
0
0.25
0.5
0.75

Fig. 3



10
1.0
0.1
0.01
0
0.5
1.0
1.5
2.0
SU(3)
ln(3g$_0^2$)
Λ_0=(5.0±1.5)x10^{-3} $\sqrt{K}$
X(I,I)
I=1
I=2
I=3
1/g$_0^2$

Fig. 4

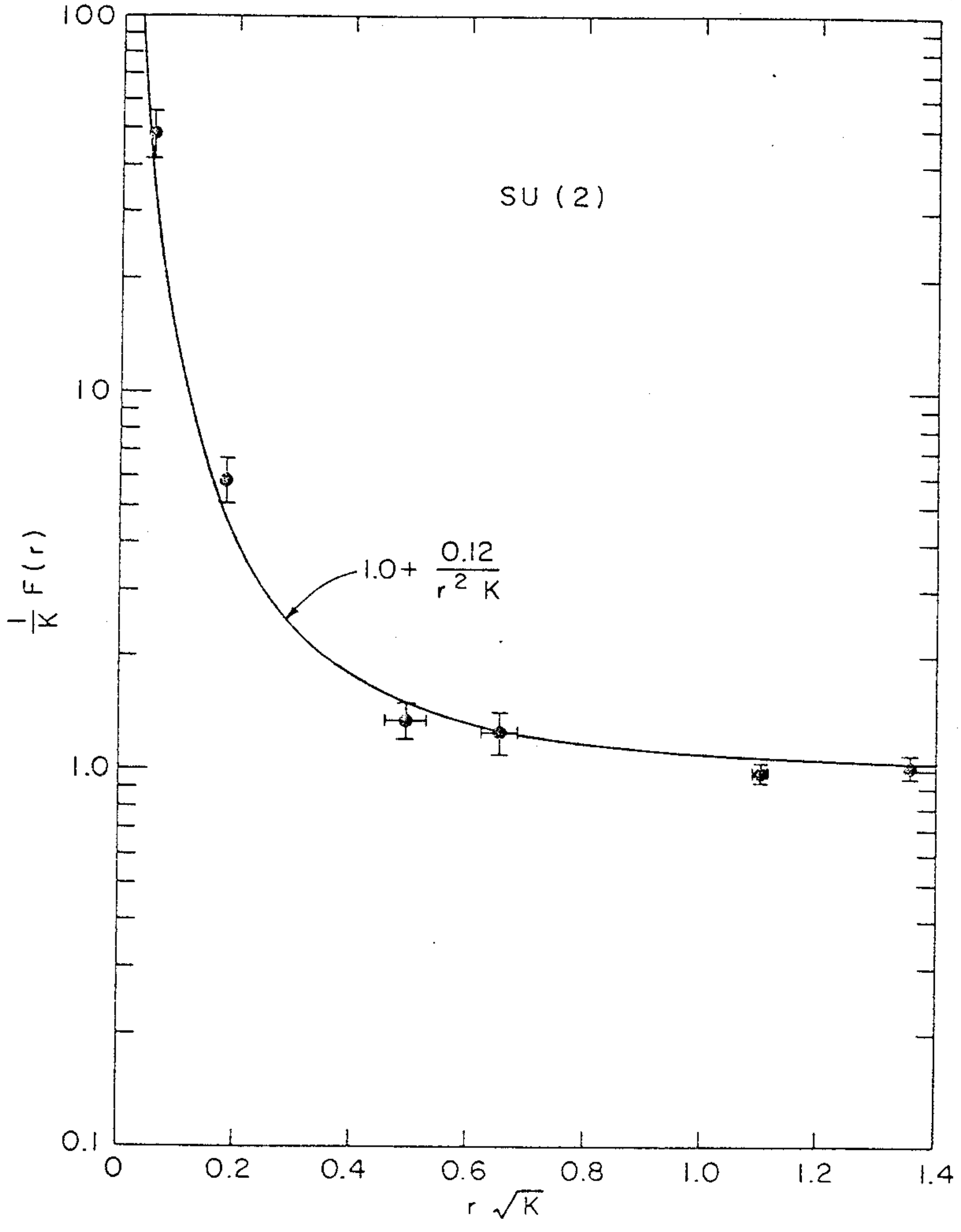

100
10
1.0
0.1
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
$r\sqrt{K}$
$\frac{1}{K}F(r)$
SU (2)
$1.0 + \frac{0.12}{r^2 K}$

Fig. 5