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Variant actions and phase structure in lattice gauge theory

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We study a simple generalization of Wilson's SU(2) lattice gauge theory. In various limits the model reduces to the usual SU(2), SO(3), or Z_2 models. Using Monte Carlo techniques on a four-dimensional lattice, we follow the known SO(3) and Z_2 first-order transitions into the phase diagram. They merge at a triple point and continue together to a critical end point. The peak in the specific heat of the SU(2) model is a shadow of this nearby singularity.

Monte Carlo studies with a lattice cutoff have given strong evidence for quark confinement in the asymptotically free SU(2) and SU(3) gauge theories. Until quite recently the lore was that lattice gauge theories based on non-Abelian gauge groups will not, in four dimensions, show any phase transitions separating a strong-coupling confining phase from the perturbative weak-coupling domain of the continuum limit. However, discoveries of unexpected transitions with the gauge groups SO(3), SU(4), and SU(5) have clouded this issue.[1,2]

The action used with a lattice cutoff (or indeed with any regulator) is highly nonunique. Wilson's formulation[3] is particularly elegant and thus has dominated research. As long as physics in the continuum limit is itself unique, the choice of lattice action is a matter of taste. However, when the lattice spacing is not small, variation of the action can modify the phase structure of the system. Indeed, the SO(3) and SU(5) transitions may be artifacts of the simple Wilson action. The mere existence of a phase transition does not necessarily imply a loss of confinement. A more general lattice Lagrangian may permit continuation around bothersome singularities.

With this motivation, we have studied a simple generalization of Wilson's model for the gauge group SU(2). We find that it is indeed possible to introduce a spurious transition in this theory. Our action has a two-parameter coupling space in which a critical point lies near to and is responsible for the rapid crossover from strong- to weak-coupling behavior in the conventional theory.

To stay as close as possible to the standard lattice gauge theory, we use the variables U_{ij} which are elements of the gauge group and are associated with the nearest-neighbor bonds $\{(i,j)\}$ of a four-dimensional simple hypercubic lattice. We also follow Wilson in keeping the action a function only of plaquette variables. Thus we restict

$$S=\sum_{\Box} S_{\Box} \,, \tag{1}$$

where the sum extends over all plaquettes $\Box$ and $S_{\Box}$ is a function only of $U_{\Box}$, an ordered product of the four group elements about the square $\Box$. Gauge invariance leads us to assume that $S_{\Box}$ is a real class function of $U_{\Box}$. Any such function can be expanded in characters

$$S_{\Box}=\sum_{R} \beta_R \operatorname{Re} \operatorname{Tr}_R(U_{\Box}) \,, \tag{2}$$

where the sum is over all representations R of the group, Tr_R is the trace of $U_{\Box}$ expressed in the given representation, and β_R are arbitrary coefficients. We throw away the imaginary part to keep the action real. The usual Wilson action keeps only the fundamental representation in this sum. Recently Manton[4] has suggested an alternative action where $S_{\Box}$ is the square of the distance of $U_{\Box}$ from the identity in the group manifold. This in general involves all representations and has been studied with Monte Carlo methods.[5]

In this paper we consider a two-parameter lattice action obtained by considering only the fundamental and adjoint representations in Eq. (2). For SU(2) we define our normalization such that

$$S_{\Box}=\beta[1-\operatorname{Tr}(U_{\Box})/2] +\beta_A[1-\operatorname{Tr}_A(U_{\Box})/3] \,, \tag{3}$$

where Tr without a subscript is taken in the twodimensional defining representation and Tr_A is in the three-dimensional adjoint representation. Note that for arbitrary SU(N) the adjoint trace is easily found using the identity

$$\mathrm{Tr}_A U = |\mathrm{Tr} U|^2 - 1 \ . \tag{4}$$

We insert this action into a path integral or partition function

$$Z = \int dU \exp\left[-\sum_{\Box} S_{\Box}\right] , \tag{5}$$

where the integration is over all bond variables. We shall see that this simple generalization of the Wilson action gives rise to a rich phase struture.

This model has several interesting limits. When $\beta_A = 0$ it reduces to the usual SU(2) theory. When $\beta = 0$ the action only depends on the adjoint representation, and corresponds to the standard Wilson theory based on the gauge group SO(3). This model is known[1] to have a first-order phase transition, which we confirm, at $\beta_A = 2.50 \pm 0.03$. Another interesting limit occurs on taking β_A to infinity. This drives all the plaquette variables to the identity in the adjoint representation. In the fundamental representation, each $U_{\Box}$ must then lie in the center of the group,

$$U_{\Box} \in \{\pm I\} \ . \tag{6}$$

This implies that under a gauge transformation all the bond variables can also be placed in the center, and we have the conventional Wegner Z_2 lattice gauge theory[6] at inverse temperature β. That model is known to exhibit a striking first-order phase transition at[7]

$$\beta = \tfrac{1}{2}\ln(1+\sqrt{2}) = 0.44 \ldots \ . \tag{7}$$

Thus at the outset we know that our model must possess a nontrivial phase diagram, with first-order lines entering from $\beta = 0.44$, $\beta_A = \infty$ and $\beta = 0$, $\beta_A = 2.5$.

We have used Monte Carlo simulation to follow these lines into the (β, β_A) plane. Our algorithm follows Metropolis *et al.*[8] Each link variable in turn is tentatively multiplied by a group element randomly selected from a table of twenty. This tentative change is accepted if a random number uniformly selected in the interval (0,1) is larger than the exponential of the change in the action. This satisfies the detailed balance requirement which ensures that an arbitrary ensemble of configurations will be brought closer to the Boltzmann distribution defined by the exponentiated action. One Monte Carlo iteration consists of applying this algorithm ten times to one link and then similarly touching all the other links of the lattice. The elements in the table were randomly selected with a coupling-dependent weighting towards the identity. A β_A-dependent fraction of the elements was selected near $-I$ to assist convergence when the Z_2 structure of the action is important. A new table was generated after each sweep through the lattice. Our boundary conditions were always periodic and no gauge fixing was imposed.

To check our results, we also studied our action for the discrete subgroup of SU(2) defined by the symmetries of an icosahedron.[9] For small β_A an extra transition due to the discrete nature of this subgroup was well separated from the interesting structures in the SU(2) model. As one expects from a simple two-state model for this "discreteness" transition, the critical couplings (β^C, β_A^C) for $\beta_A \lesssim 2$ lie on an approximately straight line well fit by

$$\beta^C = 6 - 2.4\beta_A^C \ . \tag{8}$$

It is clear from this that when β_A is of order 2-3, the discrete approximation strongly affects the phase structure. In this region we must treat the model using the full SU(2).

To monitor the behavior of this model, we measure separately the expectation of the two terms in the action. Thus we define

$$P = \langle 1 - \mathrm{Tr}(U_{\Box})/2 \rangle \ , \tag{9}$$

$$P_A = \langle 1 - \mathrm{Tr}_A(U_{\Box})/3 \rangle \ . \tag{10}$$

For a totally ordered lattice both these vanish whereas for random U_{ij} they both equal unity. Wherever $\beta = 0$ the Z_2 symmetry of the adjoint action gives $P = 1$ for any β_A. At a few values of β, β_A we also measured rectangular Wilson loops, the expectation of the trace of an ordered product of U_{ij} around a rectangle. These were measured in both the fundamental and adjoint representation.

In Fig. 1(a) we show a thermal cycle in β_A for $\beta = 0$. The lattice was 5^4 sites in size and each point represents P_A after thirty iterations at the given β_A. The crosses represent cooling of the lattice and the circles represent heating. Note the hysteresis effect due to the first-order phase transition in the SO(3) model. In Fig. 1(b) we show runs of 100 iterations starting both random and ordered at $\beta_A = 2.5$. Figure 1 represents an independent confirmation of the results in Ref. (1).

Figure 2 displays a similar thermal cycle in β with β_A fixed at 3.0. Here the order parameter P

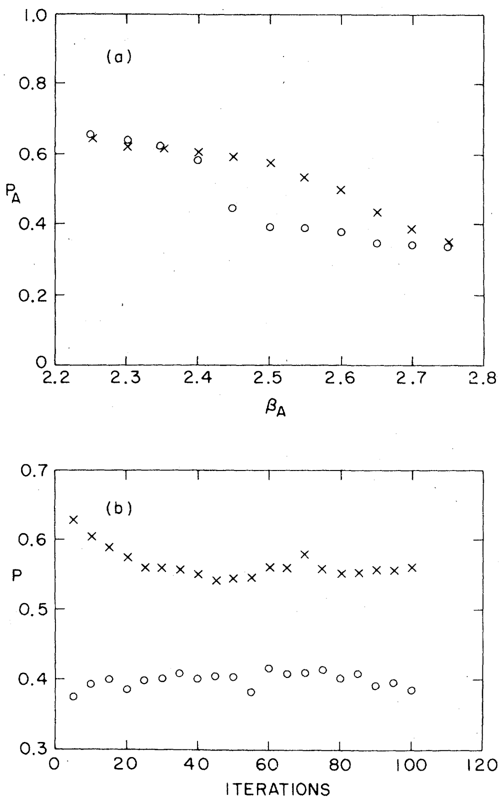

FIG. 1. (a) Thermal cycle in the SO(3) limit. Crosses represent cooling and circles represent heating. (b) Evolution of random and ordered configurations at $\beta_A=2.5$, $\beta=0$.

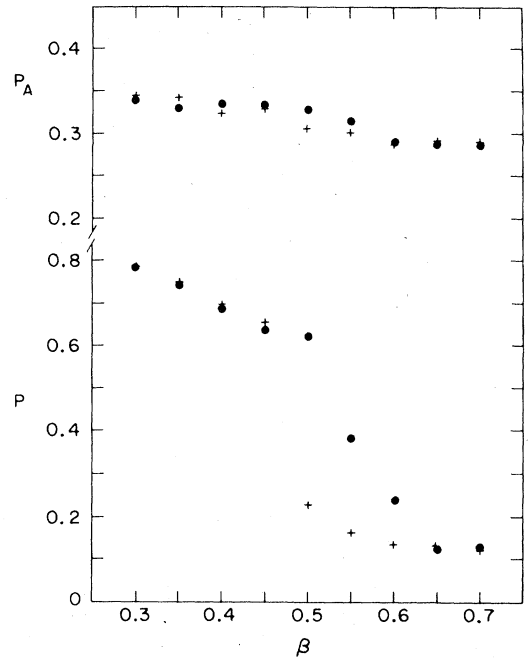

FIG. 2. Thermal cycle in β at $\beta_A=3.0$.

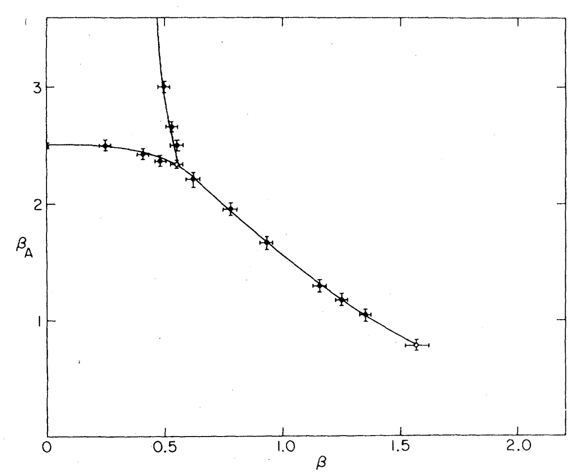

FIG. 3. The full phase diagram. The open circles represent the location of the triple point and the critical point. The solid circles trace out the first-order transition lines. The solid curves are drawn to guide the eye.

clearly shows the Z_2 transition. At this finite β_A, the transition has moved from the value in Eq. (7) to $\beta=0.50\pm0.02$. This shift to larger β is expected because decreasing β_A allows the system to be more disordered. Note that the transition is nearly unobservable in P_A.

By performing several similar thermal cycles on a 4^4 lattice we have followed both of these transitions further into the (β,β_A) plane and have obtained the phase diagram shown in Fig. 3. The Z_2 and SO(3) transitions meet at a triple point at $\beta=0.55\pm0.03$, $\beta_A=2.34\pm0.03$. Figure 4 shows three runs of one hundred iterations on a 5^4 lattice at the triple point. The three initial conditions were (1) ordered with every $U_{ij}=I$ (solid circles); (2) with each U_{ij} selected totally randomly from SU(2) (crosses); and (3) with each U_{ij} chosen randomly from $Z_2=\{\pm1\}$ (open circles). The system has three distinct stable phases at this point in coupling space.

The third first-order line emerging from this triple point extends towards smaller β_A but terminates at a critical end point before reaching the β axis. By extrapolating the latent heats in P and in P_A to vanishing values, we quote $\beta=1.48\pm0.05$, $\beta_A=0.90\pm0.03$ as the coordinates of this new crit-

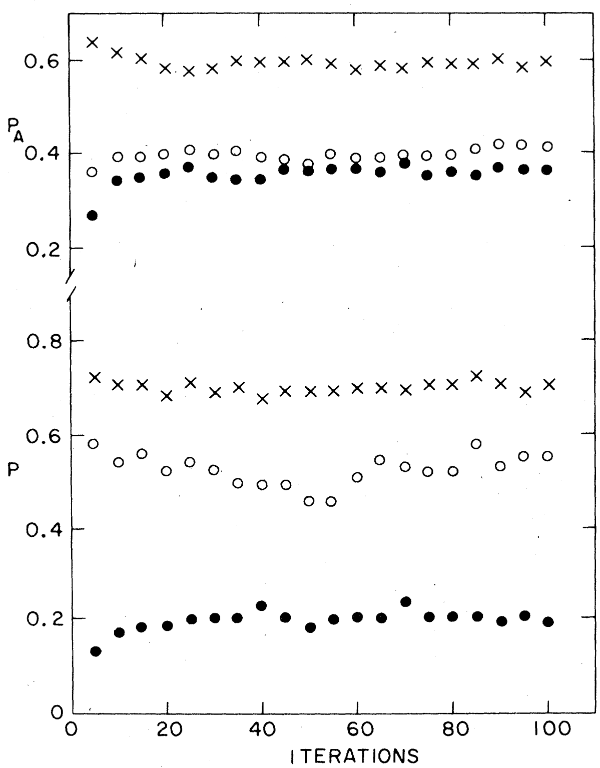

FIG. 4. Three Monte Carlo runs at the triple point.

ical point in SU(2) lattice gauge theory.

The conventional SU(2) theory exhibits a narrow but smooth peak in its specific heat[10] at $\beta=2.2$. This is directly at a naive extrapolation of the above first-order transition line to the β axis. Thus this peak may be regarded as a remnant of that transition, a shadow of the nearby critical point. Our interpretation is undoubtedly not unique as other generalizations of the theory may also find interesting structure. This picture is consistent with and indeed supports the absence of a real singularity in the standard Wilson theory.

Presumably the continum limit of the SU(2) theory is unique for physical observables. The connection between the bare field-theoretical coupling constant and our parameters β and β_A follow from an expansion of the action in powers of the cutoff

$$g_0{}^{-2}=\beta/4+2\beta_A/3 \ . \tag{11}$$

The continuum limit in this asymptotically free theory requires taking $g_0{}^2$ to zero, but this can be done along many paths in the (β,β_A) plane. Previous work has concentrated on the trajectory $\beta_A=0$, $\beta\to\infty$. Along that path no singularities are encountered, and thus confinement, present in strong coupling, should persisit into the weak-coupling domain. However, an equally justified path would be, for example, $\beta=\beta_A\to\infty$. In this case a first-order phase transition occurs. Because one can continue around it in our large-coupling-constant space, this transition is not deconfining and is simply an artifact of the action choice. The recently discovered first-order transition in SU(5) lattice gauge theory may be of a similar nature. This can be tested by adding a term with a negative β_A to the action.

To test whether physical observables are indeed independent of the path taken for a continuum limit, we measured Wilson loops on an 8^4 lattice at weak coupling for several values of β_A. The Wilson loop by itself is not an observable because of ultraviolet divergences associated with its sharp perimeter and corners. However it has been argued that ratios of loops with identical perimeters and number of corners but different shapes should be finite in the continuum limit. With this motivation, we constructed the quantities

$$R(I,J,K,L)=\frac{W(I,J)W(K,L)}{W(I,L)W(J,K)} \ , \tag{12}$$

where $W(I,J)$ is the Wilson loop of dimensions I by J in lattice units. Wishing to compare points which give similar physics, we searched in β at fixed β_A for the points where $R(2,2,3,3,)$ had the fixed values 0.87 and 0.93. This gave rise to the points in the (β,β_A) plane shown in Fig. 5. In this figure we also show the terminating first-order line discussed above and the large-β transition from the discrete approximation. The dashed lines represent contours of constant bare charge from Eq. (11). If physics is indeed similar at all these points, all ratios R of Eq. (12) should match. In Fig. 6 we show various such ratios as functions of β_A at the $R(2,2,3,3)\approx 0.87$ points from Fig. 5. To avoid clutter we have not included error bars, which are comparable to the scatter along the various curves. The comparison is quite good considering that finite cutoff corrections are ignored. We remark that if individual loops are compared without taking ratios as in Eq. (12), their values are not constant along these contours.

Note that in this comparison the bare charge is not a constant. In Fig. 5 we varied $g_0{}^2$ from less than unity to nearly 4 while holding $R(2,2,3,3)$ fixed. This variation is permissible as the bare charge is unobservable and depends on prescription. This dependence can be characterized with

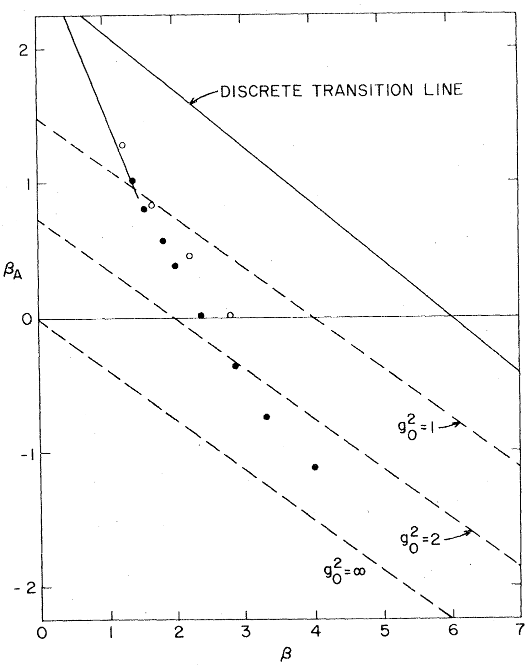

FIG. 5. Points of constant "physics." Solid circles represent $R(2,2,3,3)=0.87$, open circles represent $R(2,2,3,3)=0.93$.

an asymptotic freedom scale Λ_0 which depends on β_A. The quantity Λ_0 is an integration constant of the renormalization-group equation and is defined as the lattice spacing and g_0 became small by

$$\Lambda_0=\frac{1}{a}\left[\gamma_0 g_0{}^2\right]^{-\gamma_1/2\gamma_0{}^2} \times\exp[-1/(2\gamma_0 g_0{}^2)][1+O(g_0{}^2)]\ . \quad (13)$$

Here γ_0 and γ_1 are the first two coefficients in the perturbative expansion of the Gell-Mann-Low function,[11]

$$\gamma(g_0)=a\,dg_0/da=\gamma_0\,g_0{}^3+\gamma_1 g_0{}^5+O(g_0{}^7)\ . \quad (14)$$

For SU(2) we have

$$\gamma_0=\frac{11}{24\pi^2}\ , \quad (15)$$

$$\gamma_1=\frac{17}{96\pi^4}\ . \quad (16)$$

Assuming that the constant R contours follow a line of constant lattice spacing, we extract the β dependence for Λ_0 shown in Fig. 7. Note that the $R(2,2,3,3)=0.93$ and 0.87 results are consistent. Remarkably, the addition of β_A can change the lattice Λ_0 by several orders of magnitude.

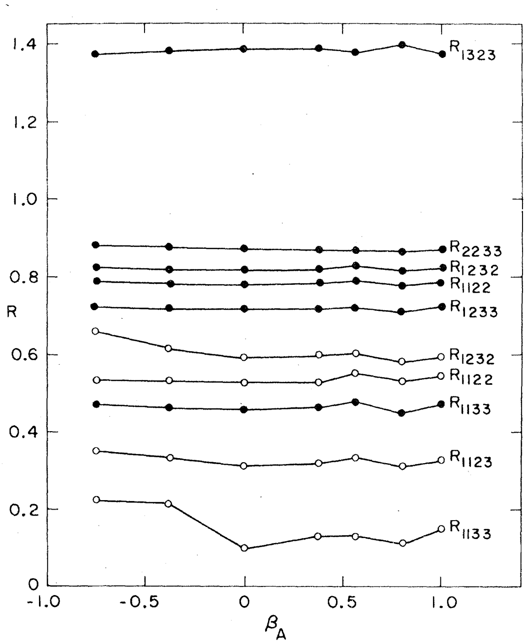

FIG. 6. Various loop ratios along the $R(2,2,3,3)=0.87$ contour. Solid circles are from loops in the fundamental representation, open circles from the adjoint.

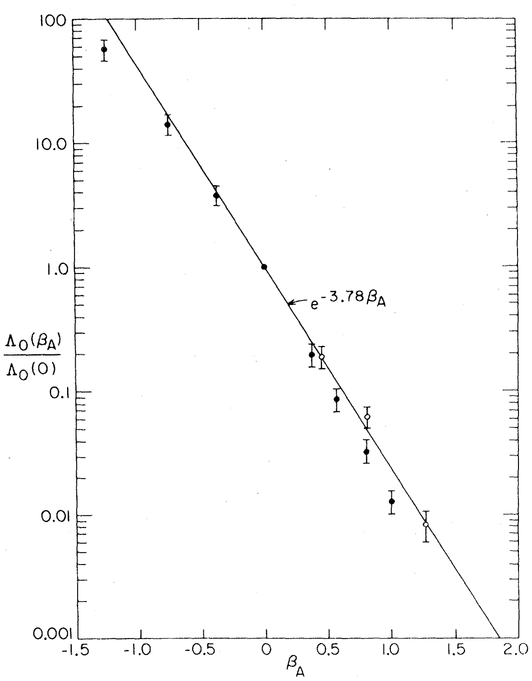

FIG. 7. The β_A dependence of the renormalization scale $\Lambda_0(\beta_A)$. The solid circles and open circles are from $R(2,2,3,3)=0.87$ and 0.93, respectively.

In conclusion, we have shown how a simple modification of the Wilson action can introduce a first-order phase transition separating the strong- and weak-coupling domains in SU(2) lattice gauge theory. The transition is not deconfining because it can be continued around in a larger coupling space. Our model has limits which select out Z_2, the center of the gauge group, and SO(3). A peculiar isolated phase at large β_A and small β is represented by the weak-coupling domain of the SO(3) model. We know of no convincing reason why this phase must be isolated. Perhaps it too is connected to the SU(2) strong-coupling phase in a yet larger coupling parameter space.[12]

ACKNOWLEDGMENT

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