Nuclear Physics B210 [FS6] (1982) 59–68

MONTE CARLO STUDIES OF WILSON LOOPS IN FOUR-DIMENSIONAL U(2) GAUGE THEORY

Michael CREUTZ

Brookhaven National Laboratory, Upton, New York 11973, USA

K.J.M. MORIARTY

Department of Mathematics, Royal Holloway College, Englefield Green, Surrey, TW20 0EX, UK

Received 6 April 1982

Wilson loops are calculated using Monte Carlo simulations for pure U(2) gauge theory on a 6^4 lattice. The loops appear to contain an area law piece in both the high and low temperature regions. The string tension is discontinuous at $\beta = \beta_c$, where β_c is the critical inverse temperature. This suggests that the first-order phase transition in U(2) gauge theory is not a deconfining phase transition. The determinant of the Wilson loop, however, extracts the U(1) part of the theory and appears to lose the area law at low temperature.

In a recent paper [1], a first-order phase transition was found in pure U(2) gauge theory in four space-time dimensions. In the present paper, we examine the Wilson loops in order to further study the nature of this U(2) transition. At low temperature the SU(2) and U(1) parts of the theory should decouple. Thus the physical parts of the loops should mimic SU(2) up to a possible renormalization of the coupling. To extract the U(1) part, we consider the determinant of the Wilson loops, as suggested by Green and Samuel [2]. This should obey a perimeter law and match pure U(1) loops at low temperature, again with a renormalization of the coupling.

We define the system in a hypercubical lattice of four euclidean space-time dimensions [3, 4]. With any link on the lattice joining nearest-neighbor lattice sites labelled by i and j, we associate a matrix U_{ij} of the U(N) gauge group such that

$$U_{ij} = \exp(i\theta_{ij})\bar{U}_{ij},$$

where $\bar{U}_{ij}$ is an $N \times N$ unitary unimodular matrix of SU(N) and θ is the angle associated with compact U(1). When the angle θ_{ij} sweeps over the interval $[0, 2\pi/N]$ and $\bar{U}_{ij}$ sweeps over SU(N), the U(N) gauge group manifold is covered. We require that the reverse path gives the inverse group element, i.e.,

$$U_{ji} = (U_{ij})^{-1}.$$

The partition function is given by

$$Z(\beta) = \int \Big(\prod_{\{i,j\}} \mathrm{d}U_{ij}\Big) \exp(-\beta S[U]),$$

where β is the inverse temperature which is related to the bare coupling constant g_0 by $\beta = 2N/g_0^2$. In the above integral, the measure is the normalized invariant Haar measure for the group. We define the action S as a sum over all unoriented plaquettes such that

$$S[U] = \sum_{\square} S_{\square} = \sum_{\square} \left(1 - \frac{1}{N} \operatorname{Re} \operatorname{Tr} U_{\square}\right),$$

where $U_{\square}$ is the product of link variables around a plaquette. We consider only a single coupling here, although since the U(1) and SU(N) pieces decouple at large β we could have separate couplings for each. This could be accomplished by adding a term proportional to the determinant of $U_{\square}$ to the action. At this point we specialize to $N = 2$. Periodic boundary conditions were used throughout our calculations. Statistical equilibrium for the lattice was achieved by means of the method of Metropolis et al. [5].

On the lattice, we define the Wilson loop [6] by the expectation value

$$W(I, J) = \tfrac{1}{2}\langle \operatorname{Re} \operatorname{Tr} U_C \rangle,$$

where C is a closed contour of rectangular dimensions I and J and U_C is a product of the link variables around the contour C. The leading-order high-temperature expansion for the Wilson loop is

$$W(I, J) = (\tfrac{1}{8}\beta)^{IJ}. \tag{1}$$

For large loops, we assume the Wilson loop behaves as

$$W \sim \exp(-A - B \cdot \text{area} - C \cdot \text{perimeter}),$$

where, the parameters A, B and C are functions of β. When this behavior applies, the string tension C is evaluated by forming the quantity

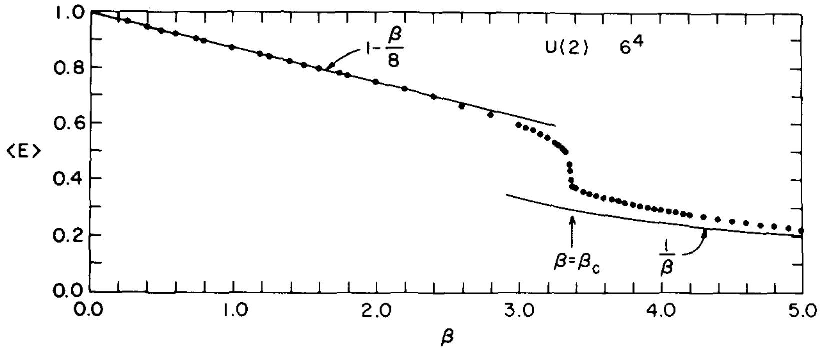
$$\chi(I, J) = -\ln \left[\frac{\langle |W(I, J)| \rangle \langle |W(I-1, J-1)| \rangle}{\langle |W(I, J-1)| \rangle \langle |W(I-1, J)| \rangle}\right]. \tag{2}$$

The leading-order high-temperature expansion for the string tension is

$$\chi(I, J) = -\ln (\tfrac{1}{8}\beta), \tag{3}$$

which should hold over a large part of the high-temperature region. Even when the area law does not dominate, the ratios in eq. (2) are useful because the divergent perimeter piece is cancelled. As argued in ref. [7] these physical ratios can serve to compare different formulations of the same theory.

Fig. 1a shows the average action per plaquette as a function of the inverse temperature on a 6^4 lattice. For these calculations, we first performed 200 iterations through the lattice each with 20 Monte Carlo upgrades per link. This equilibrates the space-time lattice. Then we averaged over the next 100 iterations through the lattice. Disordered starting lattices were usually used because it was established in



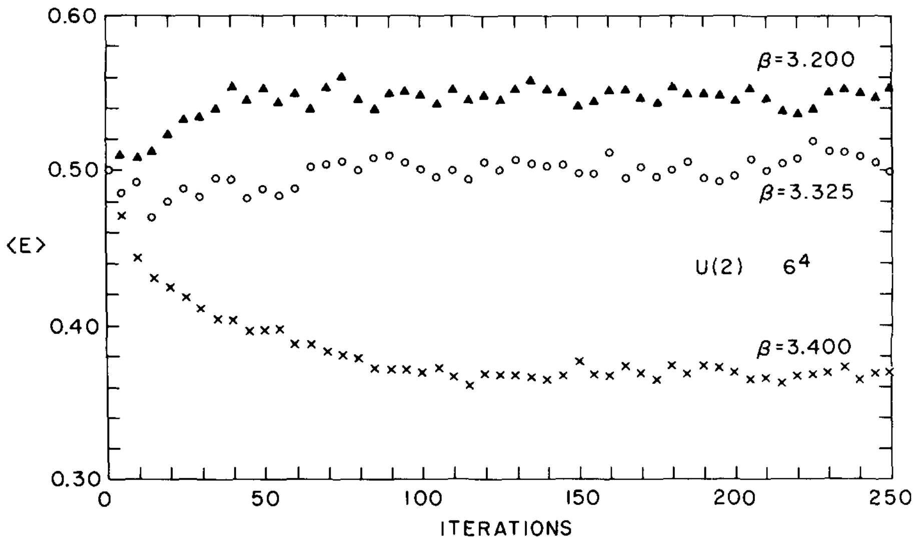


Fig. 1. (a) The average action per plaquette $\langle E\rangle$ for pure U(2) gauge theory on a 6^4 lattice as a function of the inverse temperature β. The curves represent the leading-order high and low temperature expansions of ref. [1]. (b) The evolution of the average action per plaquette $\langle E\rangle$ for pure U(2) gauge theory on a 6^4 lattice as a function of the number of iterations through the lattice for mixed phase starting lattices for various values of the inverse temperature β.

ref. [1] that 200 iterations through the lattice were sufficient to reach equilibrium. We departed from this procedure in the region $3.25 \leqslant \beta \leqslant 3.40$ where supercooling of our lattice was observed. For this range we carried out mixed phase [4] starting lattice runs where the fourth component of the euclidean lattice, the time component, was split in two with the upper half of the link variables disordered and the lower half ordered. Our results in fig. 1a agree well with fig. 1a in ref. [1]. In the present fig. 1a we also show the leading order high and low temperature expansions given in ref. [1]. In fig. 1b we show some mixed phase runs for the average action per plaquette near the critical inverse temperature. This diagram

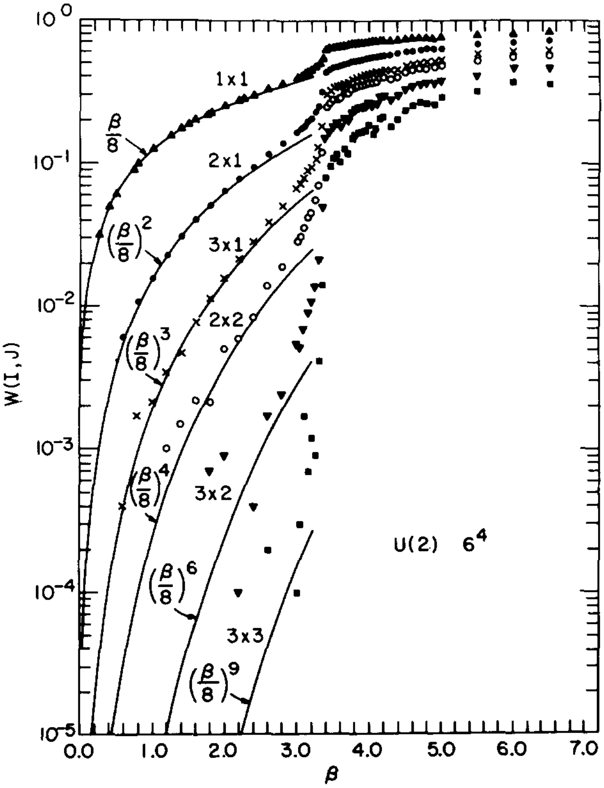


Fig. 2. The Wilson loops $W(I,J)$ for pure U(2) gauge theory on a 6^4 lattice as a function of the inverse temperature β. The full upward triangles represent $I=J=1$, the full circles represent $I=2$, $J=1$, the crosses represent $I=3$, $J=1$, the open circles represent $I=J=2$, the full downward triangles represent $I=3$, $J=2$ and the full squares represent $I=J=3$. The curves represent the leading-order high-temperature expansion of eq. (1).

shows quite clearly that we are dealing with a first-order phase transition with a critical inverse temperature of $\beta_c \simeq 3.325 \pm 0.050$. In fig. 2 we show the Wilson loops up to size 3×3. We can see from fig. 2 that eq. (1) is quite well obeyed over most of the high-temperature region.

The logarithmic ratios $\chi(I,J)$, for $(I,J)=(1,1)$, $(2,2)$, $(3,2)$ and $(3,3)$ are shown as a function of the inverse temperature β in fig. 3a. Our results agree with the leading-order high-temperature expansion of eq. (2) up to $\beta \simeq 0.8\beta_c$. As β approaches the critical inverse temperature, the string tension decreases very rapidly by about one half an order of magnitude. Fig. 3b shows an expanded view of the string tension $\chi(I,J)$ for the range of inverse temperature [3.0, 3.9]. From fig. 3b, we can see that $\chi(3,3)$ decreases at the critical inverse temperature β_c by a factor of about five. The discontinuity in the string tension is clearly visible.

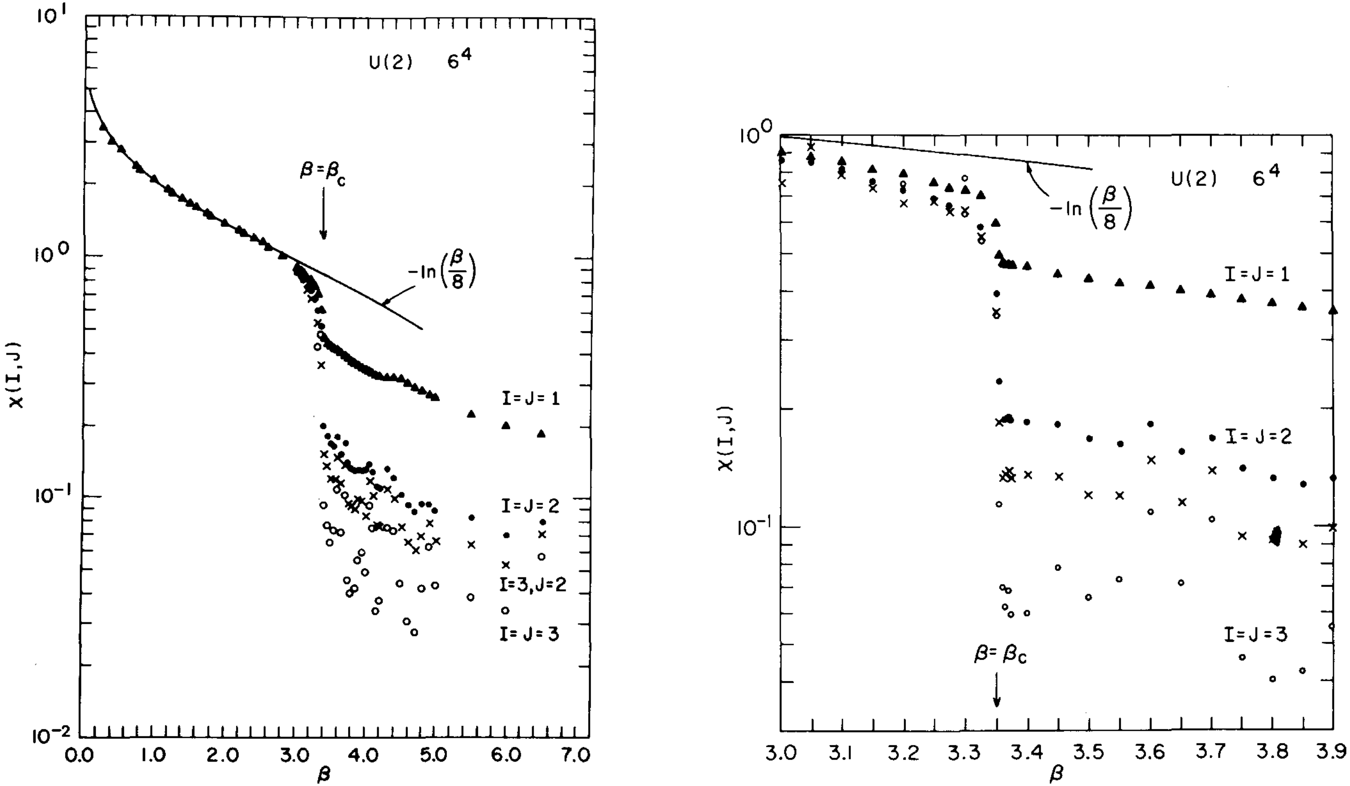


Fig. 3. The string tension $\chi(I,J)$ for pure U(2) gauge theory on a 6^4 lattice as a function of the inverse temperature β. The full upward triangles represent $I=J=1$, the full circles represent $I=J=2$, the crosses represent $I=3$, $J=2$ and the open circles represent $I=J=3$. Also shown in the diagram is the leading-order high temperature expansion of eq. (3).

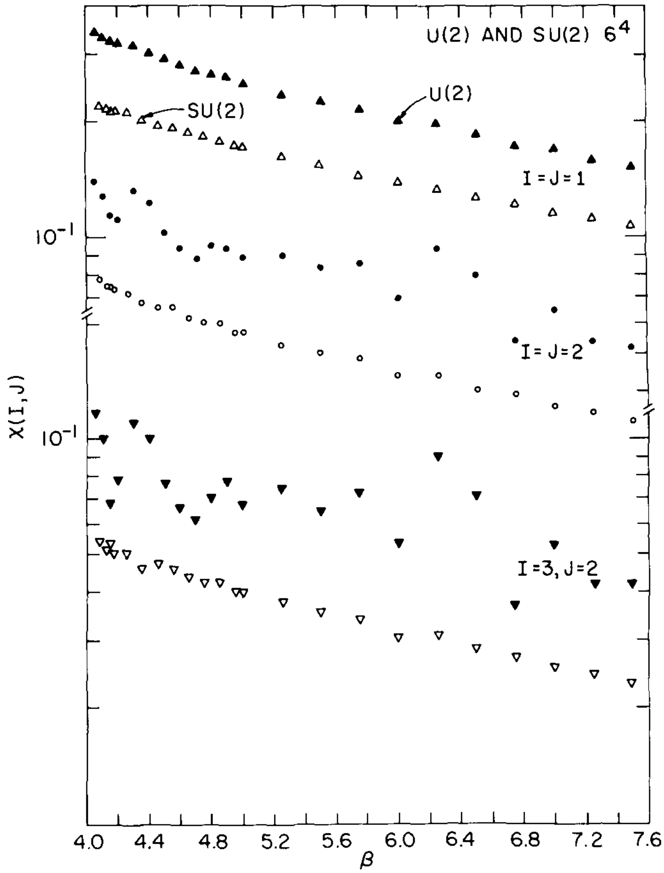


Fig. 4. A comparison of the string tension $\chi(I, J)$ as a function of the inverse temperature β for pure U(2) and SU(2) gauge theories on 6^4 lattices. The full (open) upward triangles represent $I = J = 1$, the full (open) circles represent $I = J = 2$ and the full (open) downward triangles represent $I = 3$, $J = 2$, for U(2) and SU(2), respectively.

The results for U(2) should approach the results for SU(2) [8] at large β where β may have to be renormalized by an additive constant. This is because at large β the U(1) variables should decouple and we should be left with an SU(2) theory. In fig. 4 we compare the χ ratios for these two theories. The U(2) results mimic those of SU(2) but with a shift of roughly 2 units in β.

Following a suggestion of Green and Samuel [2], we may be able to extract the U(1) string tension from the determinant of the Wilson loop. We calculated this determinant for each loop considered as a 2×2 matrix in U(2). We then average over all similar loops in each configuration to give an average denoted $\langle|\bar{W}(I, J)|\rangle$.

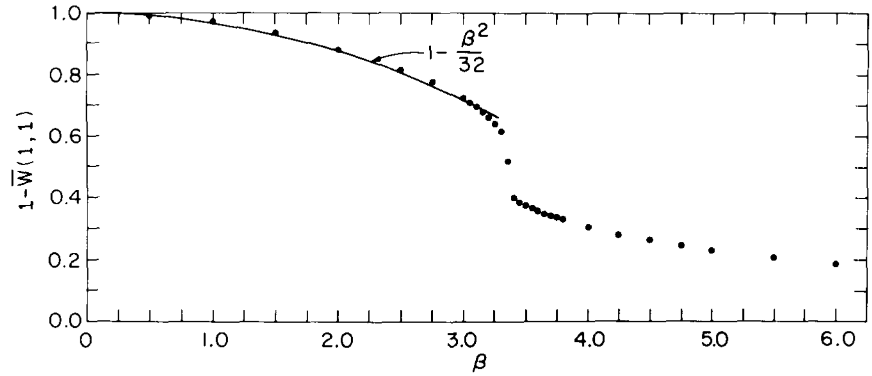


Fig. 5. The average action per plaquette $1-\bar{W}(1,1)$ for the U(1) component of pure U(2) gauge theory on a 6^4 lattice as a function of the inverse temperature β. The curve represents the leading-order high temperature expansion of eq. (5).

From these average determinants we constructed a new logarithmic ratio

$$\bar{\chi}(I,J)=-\ln\left[\frac{\langle|\bar{W}(I,J)|\rangle\langle|\bar{W}(I-1,J-1)|\rangle}{\langle|\bar{W}(I,J-1)|\rangle\langle|\bar{W}(I-1,J)|\rangle}\right]. \tag{4}$$

If the U(1) part of the theory deconfines, then this quantity should not produce an envelope in weak coupling [9, 10].

In the high-temperature region, the determinant of a wilson loop is obtained by tiling the loop with pairs of plaquettes for U(2) and N-tuplets for U(N). A straightforward analysis gives

$$\langle|\bar{W}(I,J)|\rangle=(\tfrac{1}{32}\beta^2+O(\beta^4))^{IJ}, \tag{5}$$

or

$$\bar{\chi}(I,J)=-\ln(\tfrac{1}{32}\beta^2)+O(\beta^4), \tag{6}$$

for U(2) and

$$\langle|\bar{W}(I,J)|\rangle=(\beta^N/(2^N N!N^N))^{IJ}+\cdots, \tag{7}$$

for U(N).

In fig. 5 we show the average determinant per plaquette $1-\bar{W}(1,1)$ as a function of the inverse temperature β on a 6^4 lattice. For these calculations, we used disordered starts for $\beta<3.00$, mixed phase starts for $3.00\leq\beta\leq4.00$ and ordered starts for $\beta>4.00$. As before, we first generated 200 iterations through the lattice and then averaged over the next 100 iterations through the lattice. We show the average determinants $\bar{W}(I,J)$ for $(I,J)=(1,1)$, $(2,2)$, $(3,2)$ and $(3,3)$ in fig. 6. Here we have also plotted the high-temperature expansion of eq. (5).

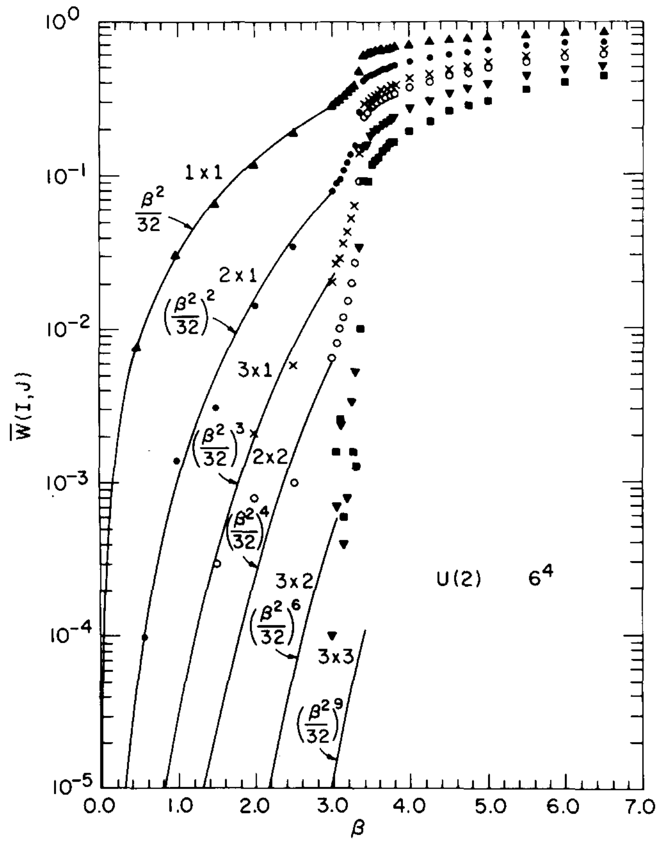


Fig. 6. The Wilson loops $\bar{W}(I, J)$ for the U(1) component of pure U(2) gauge theory on a 6^4 lattice as a function of the inverse temperature β. The full upward triangles represent $I = J = 1$, the full circles represent $I = 2$, $J = 1$, the crosses represent $I = 3$, $J = 1$, the open circles represent $I = J = 2$, the full downward triangles represent $I = 3$, $J = 2$ and the full squares represent $I = J = 3$. The curves represent the leading-order high temperature expansion of eq. (5).

In fig. 7 we show the logarithmic ratios $\bar{\chi}(I, J)$ for $(I, J) = (1, 1)$, $(2, 2)$, $(3, 2)$ and $(3, 3)$ as a function of the inverse temperature β. We can see that the high-temperature expansion of eq. (6) is well obeyed over most of the high-temperature region. In weak coupling these quantities decrease more rapidly with loop size than seen for $\chi(I, J)$ in fig. 3. In particular, the data are consistent with the quantity $\bar{\chi}(I, J)$ going to zero for increasing rectangular dimensions I and J. The determinant of the loops, however, appears to deconfine U(1) charges and suggests a massless photon at low temperature.

We would like to thank the Science and Engineering Research Council of Great Britain for the award of research grants (Grants NG-0983.9 and NG-1068.2) to

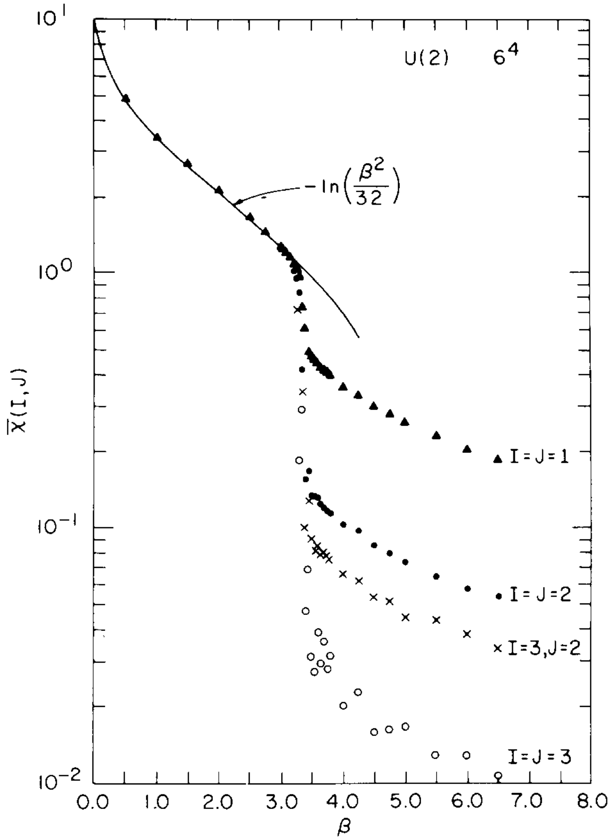


Fig. 7. The string tension $\bar{\chi}(I, J)$ for the U(1) component of pure U(2) gauge theory on a 6^4 lattice as a function of the inverse temperature β. The full upward triangles represent $I = J = 1$, the full circles represent $I = J = 2$, the crosses represent $I = 3$, $J = 2$ and the open circles represent $I = J = 3$. Also shown in the diagram is the leading-order high temperature expansion of eq. (6).

buy time on the CRAY-1 computer at Daresbury Laboratory where part of this calculation was carried out and Dr. Stuart Samuel for useful conversations and valuable correspondence.

References

[1] M. Creutz and K.J.M. Moriarty, Phys. Rev. D25 (1982) 610
[2] F. Green and S. Samuel, Phys. Lett. 103B (1981) 48;
S. Samuel, Calculating the large N phase transition in lattice gauge theories, Columbia University preprint, CU-TP-216 (October, 1981) and references therein
[3] M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. Lett. 42 (1979) 1390
[4] M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. D20 (1979) 1915
[5] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller and E. Teller, J. Chem. Phys. 21 (1953) 1087

[6] K.G. Wilson, Phys. Rev. D10 (1974) 2445
[7] G. Bhanot and M. Creutz, Phys. Rev. D24 (1981) 3212
[8] M. Creutz, Phys. Rev. D21 (1980) 2308
[9] K.J.M. Moriarty, Phys. Rev. D25 (1982) 2185
[10] K.J.M. Moriarty, Monte Carlo study of the string tension of compact U(1) gauge theory in four dimensions, Royal Holloway College preprint (November, 1981)