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# GENERALIZED ACTIONS IN $Z_p$ LATTICE GAUGE THEORY*

Michael CREUTZ and Masanori OKAWA

*Brookhaven National Laboratory, Upton, NY 11973, USA*

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In $Z_p$ lattice gauge theory we generalize the Wilson action to include all group representations. We review the implications of duality for these models. With Monte Carlo methods, we find a rich phase structure for the cases $p = 4$ and 5.

## 1. Introduction

The lattice formulation has proven to be an extremely fruitful framework for non-perturbative studies of gauge fields. Via Monte Carlo calculations with the lattice, we now have rather compelling evidence that the asymptotically free non-abelian gauge theory of quarks and gluons possesses the phenomenon of confinement. Observable free particles are bound states in the singlet representation of the gauge group. This is an essential feature for any potential theory of the strong interactions.

As with any ultraviolet regulator, the details of the lattice theory have considerable flexibility. Only in the continuum limit should we recover the presumably unique theory of the strong interactions. The original Wilson action [1], being conceptually quite simple, has dominated most research in lattice gauge theory. Nevertheless, recent studies have shown how modifications of the model can produce interesting changes in the resulting phase structure [2]. As these should be artifices of the formulation, they only occur when the lattice spacing is not small, but rather of a typical hadronic scale.

Once formulated on a lattice, gauge theories are readily extended to discrete gauge groups. These toy models have often provided a useful testing ground for techniques in lattice gauge theory. The cyclic groups $Z_p$ have been of particular interest in that they have a non-trivial phase structure and for large $p$ they approach the U(1) lattice gauge theory. Understanding the latter is essential because electrodynamics is known not to confine and is, after all, the prototype for all gauge theories.

In this paper we study the four-dimensional $Z_p$ models with a simple generalization of the Wilson theory. We keep the action as a sum over functions of single

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plaquette variables, but we consider more general class functions than simply the trace of the corresponding fundamental group matrix. In general, a class function may be expanded in characters, i.e., traces in various representations of the group. A generalization of SU(2) including only two terms in the character expansion [2] gave a rich structure interpolating between the gauge models SU(2), SO(3) and $Z_2$.

The group $Z_p$ has only a finite number of inequivalent representations. Thus the most general plaquette action depends on a finite number of parameters. Drouffe and Cardy have each discussed the duality properties of the corresponding models [3]. Recently Edgar [4] has investigated the gauge models with Monte Carlo methods and our paper should be considered an extension of that work to a wider range of couplings. In the process we have found several new transitions.

Historically, duality has served as a useful tool in the study of the $Z_p$ gauge models as well as two-dimensional spin systems. The $Z_2$, $Z_3$ and $Z_4$ theories in four dimensions are self-dual, giving rise to exact expressions for the temperatures of their phase transitions [5]. For $p > 5$ the Villain action, a slight modification of Wilson's, maintains self-duality [6]. However, here the models have two transitions dual to each other [7, 8]. With the most general action we will see some transitions on self-dual surfaces and others dual to each other. We will show how the value of $N$ at which the single transition of the low-$N$ models splits into two depends on the action. In particular, we find an intermediate Coulomb phase for $Z_4$ as well as $Z_5$ actions with only a single transition.

In the next section we will define the general model and the $[\frac{1}{2}p]$ parameters on which it depends. We will discuss various one parameter limiting cases. In sect. 3 we will review the implications of duality for the models. We find interesting constraints on the order parameters of the system when measured on the self-dual surfaces. In sect. 4 we present Monte Carlo results on several of the systems.

## 2. The generalized $Z_p$ action

As in the usual formulation of lattice gauge theory, our variables are elements of the gauge group. One such element is associated with every nearest neighbor bond on a four-dimensional periodic hypercubic lattice of $N^4$ sites. Labeling two neighboring sites by $i$ and $j$, we have an element of $Z_p$

$$U_{ij} = (U_{ji})^{-1} \in Z_p = \{\exp(2\pi ik/p) | k = 0, \ldots, p-1\}. \tag{2.1}$$

The path integral or partition function is

$$Z = \sum_{U \in Z_p} e^{S(U)}, \tag{2.2}$$

where each bond variable is summed over the group. Again as in the usual theory, the action is a sum over all elementary squares or plaquettes on the lattice

$$S(U) = \sum_{\square} S_{\square}(U_{\square}). \tag{2.3}$$

Here $U_{\square}$ is the product of the four group elements encountered in a circumnavigation of the respective plaquette. So that the orientation of the plaquette be irrelevant, we require

$$S_{\square}(U_{\square}) = S_{\square}(U_{\square}^{-1}) . \tag{2.4}$$

To interpret $e^S$ as a Boltzmann weight we restrict $S_{\square}$ to be real. For a general gauge group one usually requires $S_{\square}$ to be a class function; however, for an abelian group such as considered here, all functions are class functions.

With any gauge group the general plaquette action may be expanded in characters

$$S_{\square}(g) = \sum_n \beta_n \chi_n(g) , \tag{2.5}$$

where $\chi_n$ is the trace of $U_{\square}$ in the $n$th irreducible representation of the group. For $Z_p$ there are precisely $p$ representations, all one-dimensional and given by

$$R_n(g) = g^n , \qquad 0 \leq n \leq p-1 . \tag{2.6}$$

The representation property

$$R_n(g) R_n(g') = R_n(gg') , \tag{2.7}$$

is a trivial consequence of the abelian nature of the group. The representation combination rule is simply

$$R_n(g) R_m(g) = R_{n+m}(g) , \tag{2.8}$$

where the index is understood modulo $p$ in this and the following equation. The orthogonality of characters is

$$\frac{1}{p} \sum_g R_n(g) R_m(g) = \frac{1}{p} \sum_g g^{n+m} = \delta_{n,p-m} . \tag{2.9}$$

Thus we rewrite the parametrization of eq. (2.5):

$$S_{\square}(g) = \sum_{n=0}^{p-1} \beta_n g^n . \tag{2.10}$$

The constraint of eq. (2.4) becomes

$$\beta_{p-n} = \beta_n . \tag{2.11}$$

The parameter $\beta_0$ is a normalization, irrelevant to thermodynamics but convenient to keep. Eq. (2.9) inverts the sum in eq. (2.10):

$$\beta_n = p^{-1} \sum_g g^{-n} S_{\square}(g) . \tag{2.12}$$

For the discussion of duality it will also be convenient to expand the Boltzmann weight in characters

$$e^{S_{\square}(g)} = \sum_n x_n g^n , \tag{2.13}$$

$$x_n = p^{-1} \sum_g g^{-n} \, e^{S_{\beta_0}(g)} , \tag{2.14}$$

and eq. (2.11) becomes

$$x_{p-n} = x_n . \tag{2.15}$$

The energy shift represented by $\beta_0$ is an overall scale factor in the $x_n$. The thermodynamics depends on the $x_n$ in a projective sense.

For even $p$ the gauge group contains the element $-1$. Given any configuration of link variables we can find a corresponding one where the sign of $U_{ij}$ is changed on every plaquette. One such mapping is given in ref. [9]. This implies that the thermodynamics of the even $p$ model has a symmetry under

$$\beta_n \to (-1)^n \beta_n . \tag{2.16}$$

Often there are further symmetries in the parameter space, as we will see explicitly when we study $Z_5$.

Various contours in the general coupling space reduce to standard models. The original Wilson theory has only the coupling $\beta_1$. For $p = 2$, 3 and 4 this system has a single first-order phase transition at a point where the model is self-dual. At $p = 5$ the model develops two second-order transitions which move apart as $p$ increases further [7, 8]. One of these singularities remains in the U(1) limit of infinite $p$ and the other moves toward zero temperature inversely with the square of $p$.

The Villain [6] variation of the Wilson theory considers the one-parameter action

$$e^{S_\beta(g)} = \sum_{l=-\infty}^{\infty} e^{-\beta(\theta - 2\pi l)^2/2} , \tag{2.17}$$

where $\theta$ is defined by

$$g = e^{i\theta} . \tag{2.18}$$

This model maps onto itself under the duality transformation discussed in the next section. Another self-dual one-parameter action is the $p$ state gauge-Potts model defined by [10]

$$e^{S_\beta(g)} = \begin{cases} 1, & g = 1 , \\ e^{-\beta}, & g \neq 1 . \end{cases} \tag{2.19}$$

As a final limit consider

$$e^{S_\beta(g)} = \begin{cases} 1 , & g = 1 , \\ e^{-\beta}, & g = e^{\pm 2\pi i/p} , \\ 0 , & \text{otherwise} . \end{cases} \tag{2.20}$$

We refer to this case as the restricted $Z_p$ model, wherein all plaquette variables must be at or just next to the identity. This forces a strong degree of smoothness

on the gauge fields and for large $p$ small topological excitations are removed [11]. We will see that this restricted model displays a confined phase only for $Z_4$. This supports the idea that topological structures are essential to confinement.

## 3. Duality

Since its initial application to the Ising model [12], the concept of duality in statistical models has provided considerable insight into various phase structures. The thermodynamic function of a model may often be mapped onto those of the same or a related model with different coupling constants. Thus singularities either occur in corresponding pairs or are restricted to special "self-dual" values of the parameters.

The duality relations are particularly simple for the generalized $Z_p$ models considered here. Indeed, the inclusion of multiple couplings actually clarifies the structure of the transformation. Although we are working in the context of four-dimensional gauge theories, the duality results are identical for two-dimensional nearest neighbor spin systems. In this sense, this section merely reviews known results [3].

We begin with the character expansion in eq. (2.13),

$$e^{S_{ij}(g)} = \sum_n x_n g^n . \tag{2.13}$$

Inserting this expansion for each plaquette into the full path integral gives

$$Z = \left( \prod_{\square} \sum_{n_{\square}} \right) x_{n_{\square}} \prod_{\{ij\}} \left( \sum_{g_{ij}} \left( \prod_{\square \supset ij} g_{ij}^{n_{\square}} \right) \right), \tag{3.1}$$

where the innermost product is over the six plaquettes containing the link $ij$. The sum over the $g_{ij}$ is immediate with eq. (2.9):

$$Z = \left( \prod_{\square} \sum_{n_{\square}} \right) x_{n_{\square}} \prod_{\{ij\}} \left( p \delta_{\sum_{\square \supset ij} n_{\square}, 0} \right), \tag{3.2}$$

where the Kronecker delta is understood modulo $p$ in its indices. The factor of $p$ multiplying the $\delta$ term occurs because we have not normalized our sums over group elements. We now wish to make an appropriate change of variables and do some of the sums using the Kronecker delta functions.

The factor

$$\delta_{\sum_{\square \supset ij} n_{\square}, 0} \tag{3.3}$$

involves the six plaquettes in which link $ij$ lies. The key to simplifying this object is to go to the dual lattice. The sites of the dual lattice lie in the centers of the hypercubes of our original lattice. Labeling the new sites by $\tilde{i}$, we have for the physical coordinates

$$x_\mu(\tilde{i}) = x_\mu(i) + \tfrac{1}{2}a , \tag{3.4}$$

for all components $\mu$. To the various geometrical constructs on the original lattice such as site, link, plaquette, cube or hypercube there is a one-to-one correspondence with a hypercube, cube, plaquette, link or site, respectively, on the dual lattice. For example, dual to a link is the cube representing the common boundary of the two hypercubes dual to the ends of the link. Furthermore, the duality mapping can carry orientations if we invent a four-dimensional "left" hand rule. For example, for the dual of a link in the $+t$ direction, one has a three-dimensional cube in $x$, $y$, $z$ space and one could then orient all plaquettes on its surface to the left when viewed from its center. For other directions one should consider even permutations of the indices $xyz$ and $t$. The dual of a plaquette is another plaquette, common to the four cubes which are dual to the links of the original plaquette. The orientation is specified by the above mapping for any one of the four links.

The utility of duality in the $Z_p$ problem begins to appear with the observation that the six plaquettes in eq. (3.3), are dual to a set of six plaquettes forming the three-dimensional cube dual to the link $ij$. On the dual lattice, our partition function is a sum over integers associated with each plaquette but subject to the constraint that the sum of the variables over any three-dimensional cube is zero modulo $p$, where these plaquettes are oriented with the appropriate handed rule. Our next step is to solve this constraint.

If each of these plaquette variables were itself a modulo $p$ sum of integers on the links of the dual lattice, then this constraint would be automatic. In this case each link occurs twice, once in each orientation, in forming the cube variable. Remarkably, this solution to the constraint condition is unique up to gauge transformations. To see this, go to a completely fixed gauge, i.e., set the dual link variables to zero on a maximal tree of links which contains no closed loops. Any unfixed link must then form a unique closed loop with some set of fixed links. This link is then set, modulo $p$, to the sum of the plaquette variables on any two-dimensional surface with this loop as its boundary. The constraint condition on the cubes permits this surface to be deformed and thus assures the uniqueness of the solution procedure. If we now undo the gauge fixing, we must divide by a factor of $p^{N^4}$ gauge equivalent configurations. This process eliminates the $\delta$-function in eq. (3.2) and replaces the sum over plaquette variables with one over (dual) link variables:

$$Z = p^{3N^4} \sum_{n_{ij}} \prod x_{n_{\square}} . \tag{3.5}$$

Here $n_{\square}$ is the modulo $p$ sum of the $n_{ij}$ around the dual plaquette. We now identify $n_{\square}$ with an element of $Z_p$,

$$U_{\square} = e^{2\pi i n_{\square}/p} , \tag{3.6}$$

and do a character expansion for $x_n$,

$$x_{n_{\square}} = p^{-1/2} e^{\tilde{S}_{\square}(U_{\square})} = p^{-1/2} \sum_n \tilde{x}_n U_{\square}^n , \tag{3.7}$$

where this defines the dual action $\tilde{S}_{\square}(U_{\llcorner\lrcorner})$. In terms of these new variables we reproduce the original partition function but with a new set of parameters $\tilde{x}_n$:

$$Z(x) = Z(\tilde{x}) . \tag{3.8}$$

The connection between $x$ and $\tilde{x}$ is simply a linear transformation

$$\tilde{x}_n = A_{nm} x_m , \qquad x_n = A^{+}_{nm} \tilde{x}_m , \tag{3.9}$$

where $A$ is the unitary matrix which generates discrete Fourier transforms

$$A_{nm} = p^{-1/2} \, e^{2\pi i m n / p} . \tag{3.10}$$

This has the properties

$$A^{-1} = A^{+} , \tag{3.11}$$

$$A = A^{T} , \tag{3.12}$$

$$(A^2)_{mn} = \delta_{m,p-n} , \tag{3.13}$$

$$A^4 = I , \tag{3.14}$$

which follow directly from eq. (3.10). Eq. (3.8) is the key consequence of duality for this model.

The criterion of orientation invariance of eq. (2.15) automatically carries over to the dual variables. Note, however, that duality does not always result in a physically relevant model. If any of the $x_n$ are negative, one cannot interpret them as Boltzmann weights via eq. (3.7). For those domains of the parameter space which are dual to another sensible model, eq. (3.8) gives an interesting constraint on the singularities in the partition function. These must either appear in pairs, dual to each other, or must occur at self-dual points where $\tilde{x} = x$. We will see examples of both these cases.

We now investigate the size of the self-dual space. Thus we are interested in the unit eigenvalues of the matrix $A$,

$$x = \tilde{x} = Ax . \tag{3.15}$$

Eq. (3.14) implies that all eigenvalues of $A$ lie in the set $Z_4 = \{\pm 1, \pm i\}$. The dimension of the self-dual space is the number of independent eigenvectors of eigenvalue $+1$, and is given by

$$n_{+1} = [\tfrac{1}{4}p] + 1 , \tag{3.16}$$

where $[y]$ represents the greatest integer less than or equal to $y$. If we recall that the thermodynamics is a function of the $x_i$ only in a projective sense, then one of these dimensions is an overall scale factor. Thus the physical $Z_2$ and $Z_3$ theories should have a self-dual point, $Z_4$, $Z_5$, $Z_6$, and $Z_7$ should have a one-parameter self-dual curve, and so forth.

Differentiating the partition function gives thermodynamical quantities such as the average plaquette in various representations

$$P_n = \langle U_{\square}^n \rangle = \sum_m \frac{\partial x_m}{\partial \beta_n} F_m , \tag{3.17}$$

where

$$F = (1/6N^4) \log Z , \tag{3.18}$$

$$F_m = \frac{\partial}{\partial x_m} F . \tag{3.19}$$

From eq. (2.14) we calculate

$$\frac{\partial x_m}{\partial \beta_n} = x_{n\ m} , \tag{3.20}$$

so that

$$P_n = \sum_m x_{n-m} F_m . \tag{3.21}$$

In Fourier space this convolution becomes a product giving the alternative form

$$P_n = \sum_m A_{nm} \tilde{x}_m F_m(\tilde{x}) , \tag{3.22}$$

where

$$F_m(\tilde{x}) = \frac{\partial x_n}{\partial \tilde{x}_m} F_n(x) = A^{\dagger}_{mn} F_n(x) . \tag{3.23}$$

If we now restrict ourselves to the self-dual surface $x = \tilde{x}$, and assume that we are away from a first-order transition, then eq. (3.23) imposes a non-trivial constraint. Indeed, $F_m$ must itself be an eigenvector of $A$ with unit eigenvalues. To count the number of independent constraints first note that condition (2.11) or (2.15) requires

$$F_n = F_{p-n} , \tag{3.24}$$

which in turn implies

$$F = A^2 F . \tag{3.25}$$

Thus eq. (3.23) implies one independent constraint for each time $-1$ is an eigenvalue of $A$. This gives

$$n_{-1} = \left[ \frac{p-1}{4} \right] + 1 \tag{3.26}$$

conditions.

## 4. Specific cases

Using Monte Carlo methods we have studied the phase diagram for several of these models. We use a heat bath algorithm as described in ref. [8]. Initial studies were on a small lattice of $4^4$ sites and regions with structure were then analyzed

in more detail on lattices of up to $6^4$ sites. Boundary conditions were always periodic to minimize surface effects.

### 4.1. $Z_2$ AND $Z_3$

These models each have only one non-trivial coupling. For $Z_2$ previous analysis [8] showed a single first-order transition at the self-dual point

$$\beta_1 = \tfrac{1}{2}\log(1+\sqrt{2}) = 0.44\ldots. \tag{4.1}$$

This model has a symmetric structure at negative $\beta$.

For $Z_3$ the above analysis [8] also found a first-order transition at the self-dual point

$$\beta_1 = \beta_2 = \tfrac{1}{3}\ln(1+\sqrt{3}) = 0.335\ldots. \tag{4.2}$$

For negative $\beta$ this model in principle presents a new domain as $(-1)$ is not an element of the gauge group. This region appears to be devoid of any new phase transition. In fig. 1 we show the results of a hysteresis cycle on the negative-$\beta$ $Z_3$ model.

### 4.2. $Z_4$

At the level of $Z_4$ we obtain a model with two non-trivial couplings beyond the scale $\beta_0$:

$$S_{\mathrm{I}}(g) = \beta_0 + \beta_1(g+g^*) + \beta_2 g^2. \tag{4.3}$$

The Wilson $Z_4$ model considers $\beta_2 = 0$ and exhibits [8] a first-order transition at

$$\beta_1 = \tfrac{1}{2}\log(1+\sqrt{2}). \tag{4.4}$$

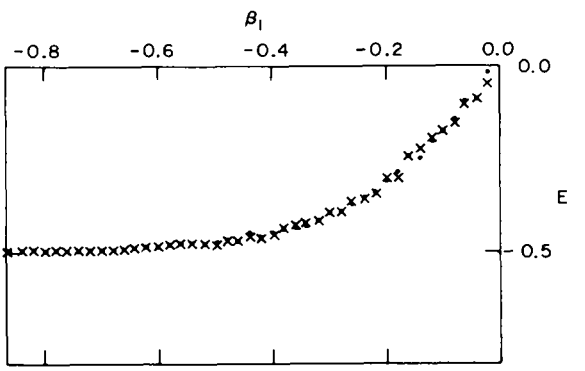


Fig. 1. A thermal cycle on the $Z_3$ model at negative coupling. The crosses represent decreasing $\beta_2$ and the circles, increasing.

The parameters $x$ are

$$x_0 = \tfrac{1}{4}e^{\beta_0}(e^{2\beta_1+\beta_2} + 2\,e^{\beta_2} + e^{-2\beta_1+\beta_2}), \tag{4.5}$$

$$x_1 = x_3 = \tfrac{1}{4}e^{\beta_0}(e^{2\beta_1+\beta_2} - e^{2\beta_1+\beta_2}), \tag{4.6}$$

$$x_2 = \tfrac{1}{4}e^{\beta_0}(e^{2\beta_1+\beta_2} - 2\,e^{-\beta_2} + e^{-2\beta_1+\beta_2}). \tag{4.7}$$

Duality reads

$$\tilde{x}_0 = \tfrac{1}{2}(x_0 + 2x_1 + x_2), \tag{4.8}$$

$$\tilde{x}_1 = \tfrac{1}{2}(x_0 - x_2), \tag{4.9}$$

$$\tilde{x}_2 = \tfrac{1}{2}(x_0 - 2x_1 + x_2). \tag{4.10}$$

Imposing self-duality, we find only a single equation,

$$x_0 = 2x_1 + x_2, \tag{4.11}$$

which in terms of $\beta_1$ and $\beta_2$ reads

$$\beta_2 = -\tfrac{1}{2}\log\sinh 2\beta_1. \tag{4.12}$$

The self-dual line starts at $\beta_2 = \infty$ when $\beta_1 = 0$ and extends toward the asymptotic form $\beta_2 = -\beta_1$ as $\beta_1 \to \infty$. We will see that only a portion of this line represents a first-order transition in the model. Elsewhere on this line we have the constraint on the expectation

$$F_0 = 2F_1 = F_2, \tag{4.13}$$

where $F_0$, $F_1$ and $F_2$ are related to the expectation of $g^n$ by

$$\langle g^0 \rangle = 1 = \tfrac{1}{4}(x_0F_0 + 2x_1F_1 + x_2F_2), \tag{4.14}$$

$$\langle g^1 \rangle = P_1 = \tfrac{1}{4}(x_0F_0 - x_2F_2), \tag{4.15}$$

$$\langle g^2 \rangle = P_2 = \tfrac{1}{4}(x_0F_0 - 2x_1F_1 + x_2F_2). \tag{4.16}$$

A little algebra yields the duality constraint

$$P_1 = \frac{1}{2\tanh\beta_1} + \tfrac{1}{2}P_2\left(2\tanh\beta_1 - \frac{1}{\tanh\beta_1}\right), \tag{4.17}$$

where $\beta_2$ is determined by eq. (4.12).

The model has several limits where it reduces to the $Z_2$ theory. The axis $\beta_1 = 0$ represents a double covering of $Z_2$. Thus we conclude that there are two first-order transitions at $\beta_1 = 0$ and $\beta_2 = \pm\frac{1}{2}\log(1+\sqrt{2})$. As we take $\beta_2$ to infinity, we drive all plaquettes to real values. Up to a possible gauge rotation, all links are then in the set $Z_2$. Thus we expect another first-order transition line to emerge from $\beta_2 = \infty$, $\beta_1 = \pm\frac{1}{4}\log(1+\sqrt{2})$. The extra factor of two is because $\beta_1 = \beta_3$ and both couplings contribute. Under duality, this transition maps into the one on the positive $\beta_2$ axis.

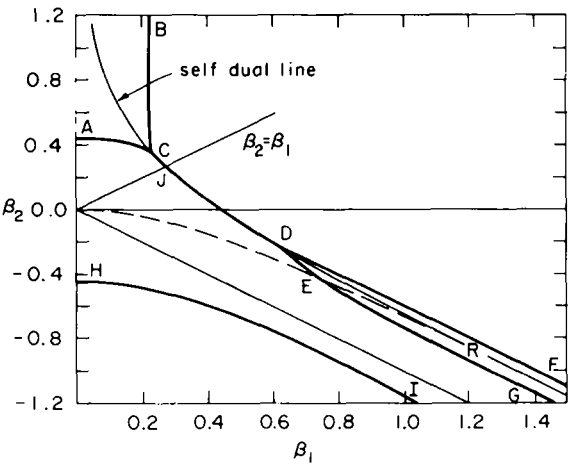


Fig. 2. The phase diagram for the generalized $Z_4$ model. The bold curves represent phase transition lines. The various other lines are discussed in the text.

The group $Z_4$ contains the element $-1$ so we have a symmetry of the phase diagram under $\beta_1 \to -\beta_1$, $\beta_2 \to \beta_2$. Because of this we need only investigate the positive $\beta_1$ half plane.

We first performed simulations of rapid thermal cycles at fixed $\beta_1$ or fixed $\beta_2$ to find the rough structures of phase diagrams. We measured both $P_1$ and $P_2$ defined in eqs. (4.15) and (4.16). Then the regions where we observed the hysteresis loops were studied carefully by the thermal cycle and by measuring $\beta_1(\beta_2)$ as a function of $P_1(P_2)$. First-order lines are distinguished from second-order transitions by measuring the latent heat at various points along the line starting from ordered and disordered initial configurations. In fig. 2 we show the phase diagram of the theory in the $\beta_1, \beta_2$ plane. The lines AC, BC, CD and HI are first-order transitions. DF and DG are second-order lines. In the graph, we also plot the self-dual line (4.12) and the line defined by

$$\beta_2 = -\tfrac{1}{2} \log \cosh 2\beta_1 \tag{4.18}$$

Eq. (4.18) is shown by the dotted line. Below this curve the dual Boltzmann weight of eq. (4.7) becomes negative and duality gives no information. The region above the self-dual line is dually related to the region between the self-dual line, the line in eq. (4.18), and the positive $\beta_2$ axis. The curve CD is on the self-dual line. The lines BC and DF are related to the lines AC and DE, respectively. EG and HI are below $\beta_2$ of eq. (4.18) so that they do not have dual partners. The first-order line CD divides into two second-order lines below the $\beta_1$ axis. The theory appears to have a spin wave phase R enclosed by these two lines. An interesting monitor in

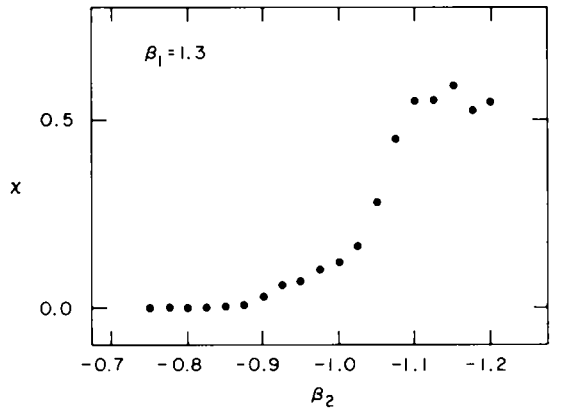


Fig. 3. The ratio $X$ of eq. (4.19) for the $Z_4$ theory is a function of $\beta_2$ at $\beta_1 = 1.3$.

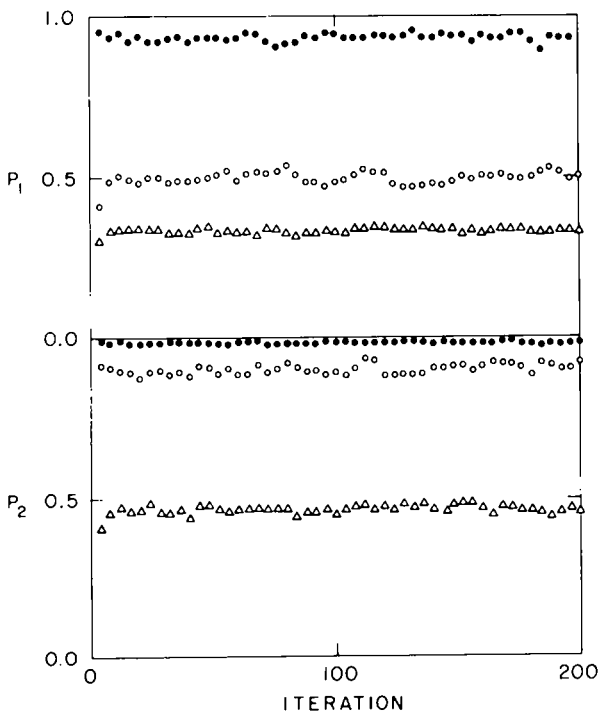


Fig. 4. Three Monte Carlo runs at the triple point C. The triangles represent an ordered start, the solid circles, a random start and the open circles, a random $Z_2$ start.

this region is the ratio of Wilson loops

$$X = \frac{W(2,2)W(1,1)}{W(2,1)W(1,2)}, \tag{4.19}$$

which is shown in fig. 3. We observe three qualitatively different behaviors of X in the regions divided by two lines DF and DG. We will see further evidence for this Coulomb phase when we discuss large $\beta$. At the triple point C we performed three runs of two hundred iterations on a $6^4$ lattice (fig. 4). The initial conditions are: (1) ordered with every $U_{ij} = 1$ (solid circles); (2) each $U_{ij}$ selected totally randomly from $Z_4$ (triangles); (3) each $U_{ij}$ chosen randomly from $Z_2$ (open circles). The system has three distinct stable phases at this point in coupling space. In particular the open circles represent the mixed phase where the fundamental action is disordered and the adjoint action is ordered. Repeating this "experiment" at self-dual points closer to the Potts line $\beta_2 = \beta_1$ indicated only two stable phases. Thus we conclude that the triple point lies away from the gauge-Potts model.

Finally we examine the constraint (4.17) on the self-dual line. Figs. 5a and b show $P_1$ and $P_2$ as functions of $\beta_1$. $\beta_2$ is determined by eq. (4.12). In fig. 5a we

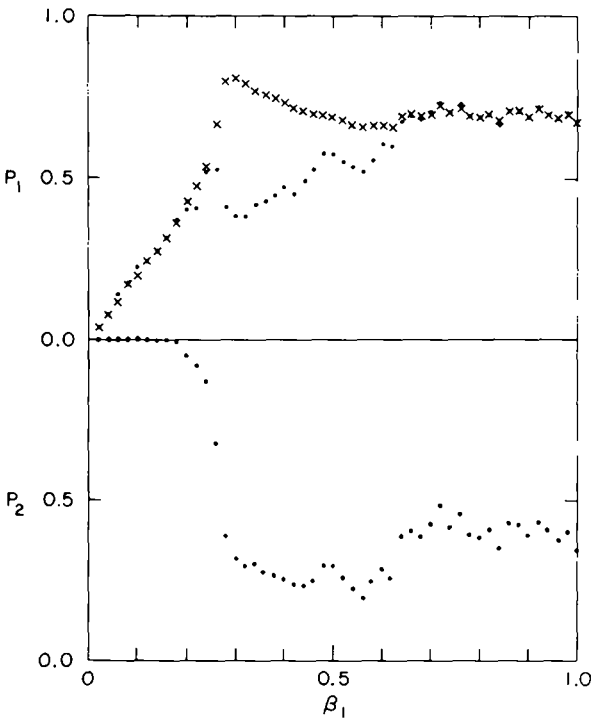


Fig. 5. Testing eq. (4.17) along the self-dual line. The crosses are the measured values for $P_1$ and the solid points the measured $P_2$ and the predicted $P_1$. Where the prediction fails we have a first-order transition on the self-dual line.

show both the experimental values of $P_1$ (solid circles) and the values calculated from the observed $P_2$ and eq. (4.17) (crosses). Except on the first-order line CD these two points coincide very well. These data further support the existence of triple point D.

### 4.3. $Z_5$

This theory also has two non-trivial couplings beyond the scale $\beta_0$:

$$S_{\square}(g) = \beta_0 + \beta_1(g + g^*) + \beta_2(g^2 + g^{*2}) . \tag{4.20}$$

Since $e^{2\pi i/5}$ and its square $e^{4\pi i/5}$ both generate the whole group, the thermodynamics of the model is symmetric under interchange of $\beta_1$ and $\beta_2$.

The Wilson $Z_5$ model ($\beta_2 = 0$) exhibits two second-order transitions near

$$\beta_1 = 1.0 \text{ and } 1.2 . \tag{4.21}$$

The parameters $x$ are

$$x_0 = \tfrac{1}{5} e^{\beta_0}[e^{2\beta_1+2\beta_2} + 2 e^{2c_1\beta_1+2c_2\beta_2} + 2 e^{2c_2\beta_1+2c_1\beta_2}] , \tag{4.22}$$

$$x_1 = \tfrac{1}{5} e^{\beta_0}[e^{2\beta_1+2\beta_2} + 2c_1 e^{2c_1\beta_1+2c_2\beta_2} + 2c_2 e^{2c_2\beta_1+2c_1\beta_2}] , \tag{4.23}$$

$$x_2 = \tfrac{1}{5} e^{\beta_0}[e^{2\beta_1+2\beta_2} + 2c_2 e^{2c_1\beta_1+2c_2\beta_2} + 2c_1 e^{2c_2\beta_1-2c_1\beta_2}] , \tag{4.24}$$

where $c_1 = \cos\frac{2}{5}\pi$ and $c_2 = \cos\frac{4}{5}\pi$. Duality reads

$$\tilde{x}_0 = \sqrt{\tfrac{1}{5}}(x_0 + 2x_1 + 2x_2) , \tag{4.25}$$

$$\tilde{x}_1 = \sqrt{\tfrac{1}{5}}(x_0 + 2c_1x_1 + 2c_2x_2) , \tag{4.26}$$

$$\tilde{x}_2 = \sqrt{\tfrac{1}{5}}(x_0 + 2c_2x_1 + 2c_1x_2) . \tag{4.27}$$

Self-duality imposes a single relation among the $x$,

$$x_0 = -2c_2(x_1 + x_2) , \tag{4.28}$$

which in terms of $\beta_1$ and $\beta_2$ reads

$$\beta_2 = -\beta_1 + \tfrac{2}{5}[\ln(1+\sqrt{5}) + \ln\cosh\tfrac{1}{2}\sqrt{5}(\beta_1 - \beta_2)] . \tag{4.29}$$

As mentioned above, the theory is symmetric under the exchange $\beta_1 \leftrightarrow \beta_2$; so, we only consider $\beta_1 > \beta_2$. The self-dual line starts out $\beta_1 = \beta_2 = \frac{1}{5}\ln(1+\sqrt{5})$ and extends toward the asymptotic form $\beta_2 = [(c_1 - 1)/(1 - c_2)]\beta_1$ as $\beta_1 \to \infty$. As in the $Z_4$ case we will see that a part of this line represents a first-order transition. The special line defined by $x_2 = 0$ reads, in terms of $\beta_1$ and $\beta_2$, as

$$\beta_2 = -\beta_1 + \tfrac{2}{5}\ln[\sqrt{5}\sinh\tfrac{1}{2}\sqrt{5}(\beta_1 - \beta_2) + \cosh\tfrac{1}{2}\sqrt{5}(\beta_1 - \beta_2)] . \tag{4.30}$$

In fig. 6 we show the result of our simulations. The lines AB and EF are first-order transitions. BC and BE appear to be second-order lines. The line EG is first-order

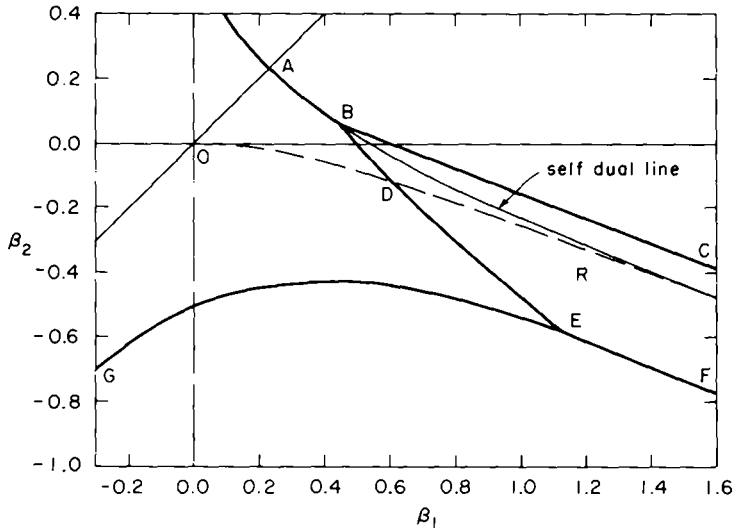


Fig. 6. The $Z_5$ phase diagram. The bold curves represent phase transition lines.

near point E but it is difficult to distinguish whether the transition remains so as we go to the left towards G. The line BE becomes difficult to detect by thermal cycle as $\beta_2$ becomes more negative. To enhance the transition we again turned to ratios of Wilson loops as in eq. (4.19). As an example we show in fig. 7 the result with fixed $\beta_2 = -0.3$. The region R is the spin wave phase.

The line AB is self-dual. Line BC is dually related to the line BD. The lines DE and FG are below the $X_2 = 0$ line so that they do not have dual partners.

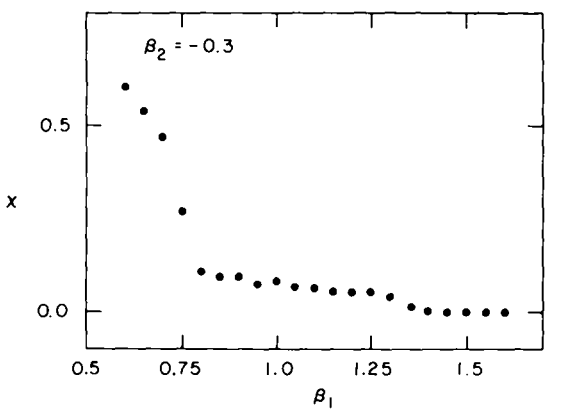


Fig. 7. The $X$ ratio of eq. (4.19) for the $Z_5$ model along the line $\beta_2 = -0.3$.

4.4. LIMITING CASES

In this subsection we consider the limiting models mentioned in sect. 2. For every $p$ there is a single line in the coupling constant space on which the theory reduces to the Potts model [eq. (2.19)]. For example, in $p=4$ and 5, this line is given by $\beta_2=\beta_1$. On this line the action $S_{\square}(g)$ with $g\neq 1$ has the same value. The gauge-Potts model was studied in ref. [10] and has one first-order transition on the positive $\beta$ axis. For $Z_4$ and $Z_5$ this transition corresponds to the points J in fig. 2 and A in fig. 6.

Next we consider the restricted model defined in eq. (2.20). To obtain this model from the generalized $Z_p$ theory it is sufficient to retain the first two non-trivial coupling constants

$$S_p(i)=\beta_1\cos\frac{2\pi}{p}i+\beta_2\cos\frac{4\pi}{p}i\,,\qquad 0\leq i\leq p-1 \tag{4.31}$$

[$\beta_i$ in eq. (4.31) differs from $\beta_i$ in eqs. (4.3) and (4.20) by a scale factor]. In eq. (4.31) we put

$$\beta_2=-\frac{1-\cos(2\pi/p)}{1-\cos(4\pi/p)}\beta_1+\beta\,, \tag{4.32}$$

and let $\beta_1\to\infty$. In this limit eq. (4.31) reduces to

$$e^{S_p(i)}=\begin{cases}1\,, & i=0\,,\\ e^{-(1-\cos(4\pi/p))\beta}\,, & i=1 \text{ and } p-1\,,\\ 0\,, & \text{otherwise}\,.\end{cases} \tag{4.33}$$

In fig. 8 we show the result of the rapid thermal cycle for $p=4$. The vertical axis is the expectation values of $P_2=\langle\cos(4\pi i/p)\rangle$. We observe three phase transitions.

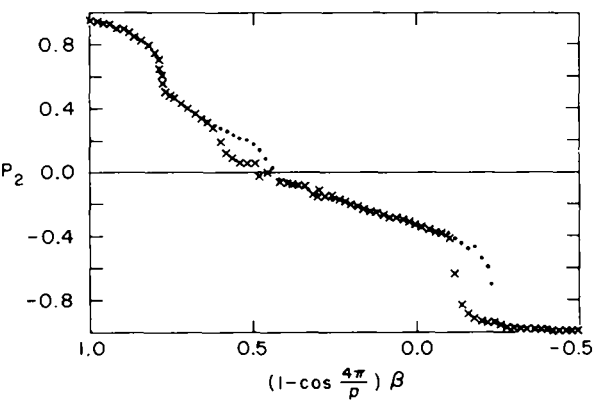


Fig. 8. A thermal cycle on the restricted $Z_4$ model discussed in the text. Note the appearance of four distinct phases.

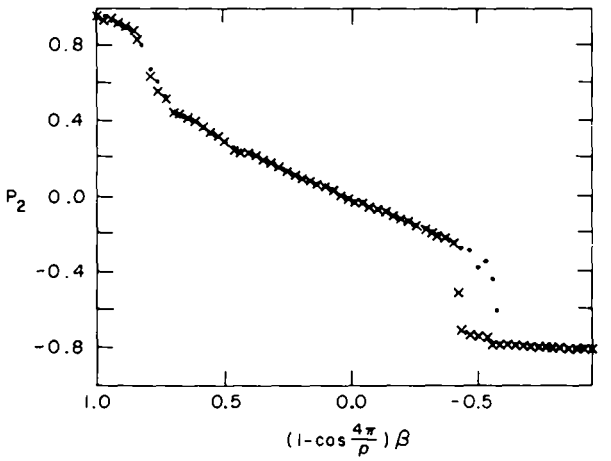


Fig. 9. A thermal cycle on the restricted $Z_5$ model. Here only three phases appear.

The first two transitions are second order and the last one is first order. This is consistent with the phase diagram shown in fig. 2. Fig. 8 further confirms the bifurcation at point D in the latter figure.

For $p \geq 5$ there are only two phase transitions. One is second order and the other one is first order. It is interesting to note that the critical coupling $\beta_c$ scales quite accurately like $\beta_c \sim 1/(1-\cos(4\pi/p))$; in other words, $(1-\cos(4\pi/p))\beta_c$ is almost constant for $p \geq 5$. In figs. 9 and 10 we show the results of $p = 5$ and 8. Note that only in the $p = 4$ case does any remnant of the confining phase persist in this restricted limit.

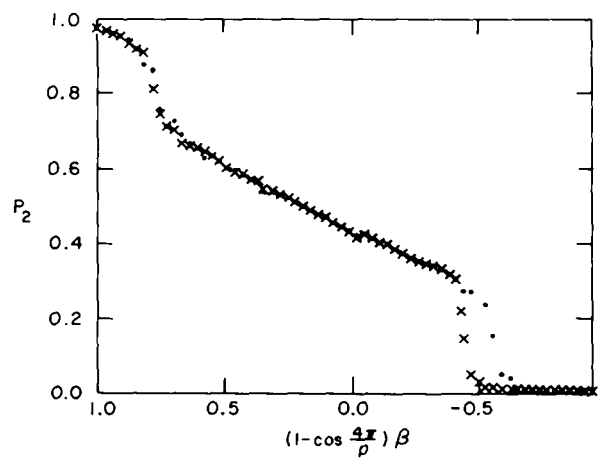


Fig. 10. A thermal cycle on the restricted $Z_8$ model. Note the similarity to fig. 9.

*Note added in proof*: After completion of this work we received preprints detailing similar results from F. Alcaraz and L. Jacobs and from M. Fukugita, T. Kaneko, and M. Kobayashi, Nucl. Phys. B215 [FS7] (1983) 289.

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