J. Phys. G: Nucl. Phys. 10 (1984) 867–873. Printed in Great Britain

The phase diagrams for SU(N)–SU(N)/Z_N, N = 3–8, gauge theories in four dimensions

R W B Ardill†, Michael Creutz‡ and K J M Moriarty§||

† Department of Mathematics, Royal Holloway College, Englefield Green, Surrey TW20 0EX, UK

‡ Brookhaven National Laboratory, Upton, New York 11973, USA

§ Department of Physics, Technion–Israel Institute of Technology, Haifa, Israel

Received 22 September 1983

Abstract. By computer simulation, we study SU(N) lattice gauge theory with an action containing couplings with the plaquette trace in both the fundamental and adjoint representations of the group. We investigate whether such an action will permit continuation around the phase transition of the Wilson theory when N exceeds four. By extrapolating this transition to where its latent heat vanishes, we locate the corresponding critical points for N up to eight.

The phase structure of lattice gauge theory has been of crucial importance to our understanding of quark confinement. The absence of phase transitions in the four-dimensional models based on SU(2) (Creutz 1979) and SU(3) (Creutz and Moriarty 1982b, Ardill *et al* 1983a) indicates that the linear interquark potential at large separations persists from the strong-coupling into the weak-coupling domain. Recently, this simple picture acquired a fascinating complication with the discovery of phase transitions in the Wilson theory when the gauge group is SU(N) with $N \geqslant 4$ (Creutz 1981, Moriarty 1981, Bohr and Moriarty 1981, Creutz and Moriarty 1982a). It was suggested by Creutz (1981) that these transitions were not deconfining, but could be continued around in a larger coupling space. It was later found that SU(2) (Halliday and Schwimmer 1981a, b, Greensite and Lautrup 1981, Bhanot and Creutz 1981) and SU(3) (Bhanot 1982) theories possessed a rich phase structure with such an extension. In this paper, we investigate larger-N models with both fundamental and adjoint couplings and find evidence that the transition with the Wilson action is indeed spurious and is easily continued around.

We work on a hypercubical lattice in four Euclidean space–time dimensions. Our action is

$$S(\beta, \beta_A, [U]) = \sum_{\square} \frac{\beta}{2N} \mathrm{Tr}_F (U_\square + U_\square^\dagger) + \sum_{\square} \frac{\beta_A}{N^2-1} |\mathrm{Tr}_F U_\square|^2 \tag{1}$$

where $U_\square$ is the product of link variables around a plaquette. Periodic boundary conditions are used throughout the calculation. The first term in equation (1) is the usual Wilson action while the second term, up to an additive constant, sums plaquette traces in the

|| Permanent address: Department of Mathematics, Royal Holloway College, Englefield Green, Surrey TW20 0EX, UK.

adjoint representation. We define our partition function by

$$Z(\beta, \beta_A) = \exp(F(\beta, \beta_A)) = \sum_{[U]} \exp(S(\beta, \beta_A, [U]))$$

and our order parameters by

$$\langle E \rangle = 1 - \frac{1}{N_p} \frac{\partial F}{\partial \beta} = 1 - \frac{1}{N} \langle \mathrm{Re\,Tr}\, U_\square \rangle \tag{2}$$

and

$$\langle E_A \rangle = \frac{N^2}{N^2-1} - \frac{1}{N_p} \frac{\partial F}{\partial \beta_A} = 1 - \frac{1}{N^2-1} \mathrm{Tr}_A\, U_\square \tag{3}$$

where F is the free energy, N_p is the number of plaquettes in the lattice and Tr_A denotes the adjoint character. In the classical continuum limit this action reduces to the conventional Yang–Mills theory with bare coupling determined by

$$g_0^{-2} = \frac{1}{2N} \beta + \frac{N}{N^2-1} \beta_A.$$

Once the system is equilibriated by the method of Metropolis *et al* (1953), we measure the average actions per plaquette $\langle E \rangle$ and $\langle E_A \rangle$ for the fundamental and adjoint representations, respectively. Further details on the calculational procedure can be found in the paper by Ardill *et al* (1983b). All our Monte-Carlo simulations were carried out on 3^4 lattices, with sampling from a table of 50 SU(N) matrices and 5 hits per link. Such a small lattice would be too small for a group as simple as, say, SU(2), but it was argued by Creutz (1981) that as N increases, the large group parameter space should partially compensate for the small number of space degrees of freedom. This is also supported by the recent work with the Eguchi–Kawai model (Eguchi and Kawai 1982; see also Bhanot *et al* 1982, Bhanot and Moriarty 1983) wherein interesting physics is obtained from a one-site lattice.

Let us consider the qualitative features of the phase diagram resulting from equation (1). Now SU(N)/Z_N theory, obtained along the line $\beta = 0$, has first-order phase transitions (Creutz and Moriarty 1982c) at $\beta_A = 6.4 \pm 0.1$, 12.0 ± 0.35, 19.5 ± 1.1 and 32.0 ± 1.0 for $N = 3, 4, 5$ and 6, respectively.

These critical points seem to scale in the variable $\beta_A/2N^2$ and we use this to predict the SU(N)/Z_N critical points for $N=7$ and $N=8$. For $\beta_A = \infty$, equation (1) becomes a Z_N gauge theory which has the critical points given in table 1 (taken from Creutz *et al* 1979). When the corresponding two-coupling system was studied for the group SU(2) by Bhanot and Creutz (1981), it was found that the Z_2 and SU(2)/Z_2 transitions entered the diagram

Table 1. The critical points for pure Z_N gauge theory taken from Creutz *et al* (1979).

N	Critical points in β
3	0.67
4	0.88
5	1.0, 1.2
6	1.0, 1.6
7	1.0, 2.1
8	1.0, 2.7

and met at a triple point, from which another first-order line emerged. These two transitions move only slightly from their initial values of β or β_A, respectively, before they meet. The third line is remarkably straight in its approach to the cross-over region of the Wilson theory. We assume that the qualitative features of this upper right quadrant of the phase diagram are similar in the higher-N cases. When N is five or more, the Z_N theory has two phase transitions separated by a Coulomb phase. We assume that these also extend into the diagram at approximately constant β. In figure 1 we show the generic form assumed for the phase diagram. We use this form to estimate the location of the triple point from which the line BC in the diagram emerges. The true triple point should lie somewhat below and to the right of this estimate; however, because of the large lever arm, this should not strongly influence our conclusions concerning the end-point C. In this paper we do not investigate the details of the region around the two triple points A and B. We also leave open the question of further structure in the negative β_A half plane.

Recently Bachas and Dashen (1982) speculated that the appearance of the spurious transition in these models may correlate with the development of new local minima of the plaquette action as a function of the gauge group. They pointed out that for SU(N) with $N > 4$ the trace of a group element is locally maximised for several elements of the group centre other than just the identity. Adding the adjoint coupling to the trace can introduce

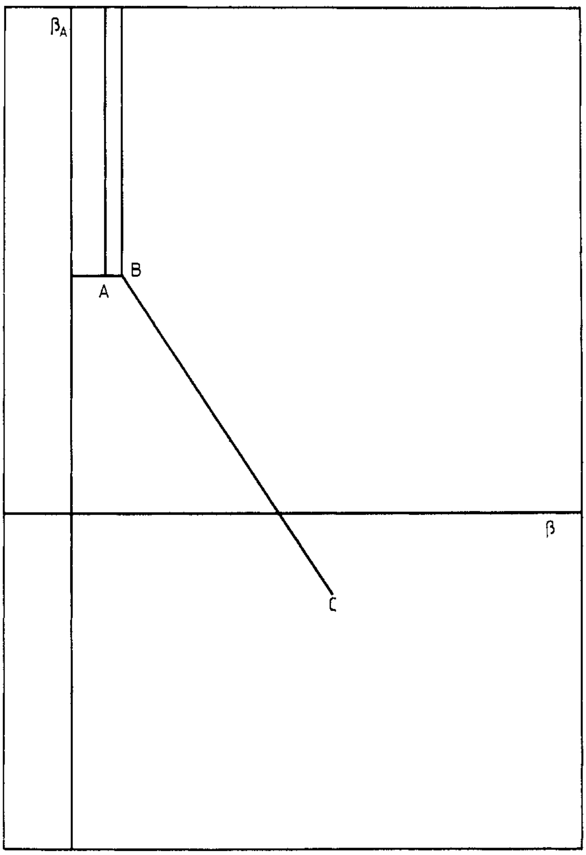

Figure 1. The SU(N)–SU(N)/Z_N phase diagram showing the Z_N, SU(N)/Z_N and SU(N) critical points, the two triple points A and B and the critical point C.

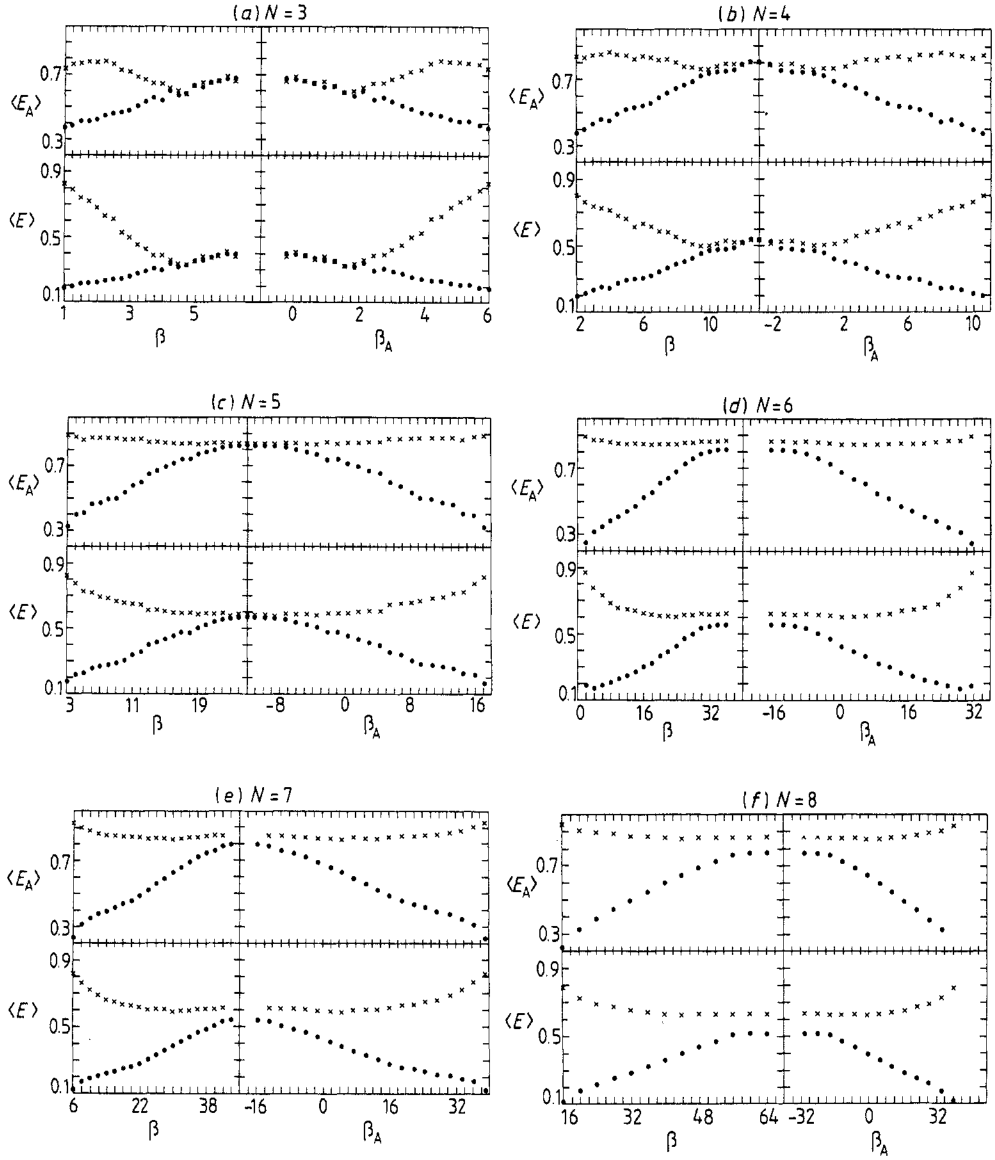

Figure 2. Equilibrium values of $\langle E \rangle$ and $\langle E_A \rangle$ starting from disordered (×) and ordered (●) configurations for various (β, β_A) values along the critical line BC for SU(N)–SU(N)/Z_N: (*a*) $N=3$; (*b*) $N=4$; (*c*) $N=5$; (*d*) $N=6$; (*e*) $N=7$; (*f*) $N=8$.

such extra minima for $N \leqslant 4$. For SU(N) one can easily verify that the line above which extra action minima occur is

$$\beta_A = -\frac{N^2-1}{2N^2}\cos\left(\frac{2\pi}{N}\right)\beta. \tag{4}$$

Note that for $N<4$ this has a positive slope, and $N=4$ is a borderline case. The end-points of the SU(2) and SU(3) transitions lie near this line (Bhanot 1982).

These arguments suggest that classical solutions of lattice gauge theory play a role in

Table 2. The critical points for SU(N)–SU(N)/Z_N gauge theory corresponding to the point C in figure 1 along with the intercept of the critical line BC with equations (4) and (5).

N	Critical points in (β, β_A)	Intercept of critical line with equation (4)	Intercept of critical line with equation (5)
3	$(4.35 \pm 0.10, 1.60 \pm 0.10)$	(5.1, 1.2)	
4	$(10.10 \pm 0.20, -0.20 \pm 0.20)$	(10.2, 0.0)	
5	$(23.0 \pm 1.0, -8.3 \pm 1.0)$	(18.6, −2.8)	(27.4, −12.6)
6	$(34.0 \pm 2.0, -14.0 \pm 2.0)$	(29.0, −7.2)	(36.6, −17.8)
7	$(46.0 \pm 3.0, -20.0 \pm 3.0)$	(41.2, −12.8)	(47.0, −21.6)
8	$(64.0 \pm 3.0, -32.0 \pm 3.0)$	(58.5, −21.4)	(66.0, −32.4)

these spurious transitions. The lowest action classical configuration has all links gauge equivalent to the identity. Consider multiplying one link by an element of the group centre which is nearest the identity. If we are above the line in equation (4), the action of this new configuration will be increased by any small perturbation of the fields. Thus we have a local minimum of the action, a classical solution in the conventional sense. A condensation of these objects may induce the new transitions.

Starting with the estimated triple point (point B in figure 1), with either an ordered or disordered start, we travelled down the straight line BC through the known transition or cross-over of the Wilson theory, continuing in equal increments of β until a value approximately on the line

$$\beta_A = -\frac{N^2-1}{2N^2}\beta, \qquad (5)$$

where the bare coupling becomes infinite, was reached.

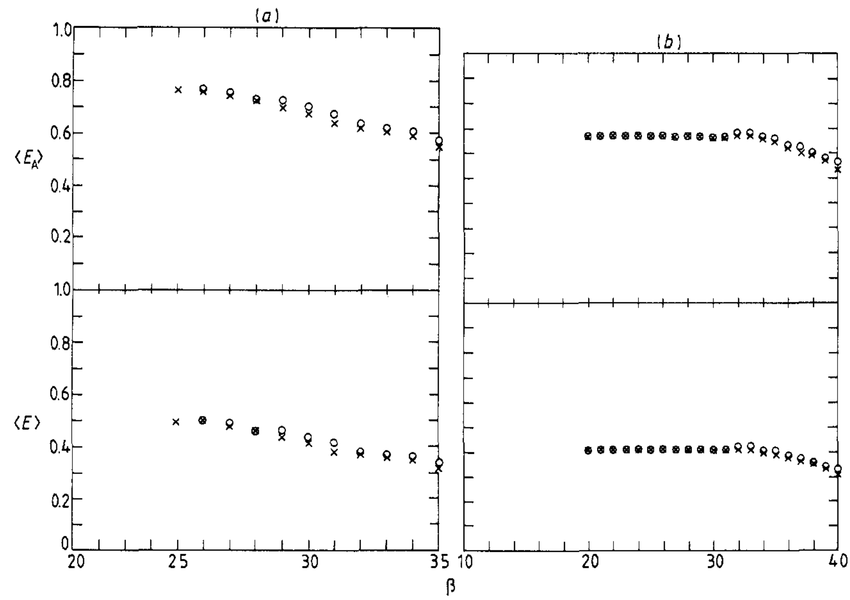

Figure 3. Hysteresis cycle for SU(5)–SU(5)/Z_5 gauge theory for fixed negative β_A: (a) $\beta_A = -11.0$; (b) $\beta_A = -14.0$. (The open circles represent the heating cycle while the crosses represent the cooling cycle.)

Figures 2(*a*), (*b*), (*c*), (*d*), (*e*) and (*f*) show the equilibrium values of $\langle E \rangle$ and $\langle E_A \rangle$ obtained with a hysteresis cycle moving towards the critical point along the line BC of figure 1. We used 50 Monte-Carlo iterations per (β, β_A) value, with an average over the last five iterations. We employed both ordered and disordered starts for our hysteresis cycles. For the graphs shown in figure 2 the $N=3$ case took about 10 min for either an ordered or disordered start and 1 h for $N=7$ and 8.

We observe that near the the critical point the ordered and disordered lines approach each other approximately linearly. This was used to estimate the critical points shown in table 2. Also given in table 2 are the points obtained by the interception of equations (4) and (5) with the critical line BC (of figure 1). From table 2 we can see that as N increases the critical end-point C moves below equation (4) and towards equation (5).

Our implementation of the algorithm of Metropolis *et al* performs badly for large β and large negative β_A; this can be seen in the disordered start lines in figures 2(*e*) and (*f*) for $N=7$ and 8, respectively, which tend to bend upwards. This is probably due to a poor optimisation of parameters on our part.

In order to confirm that we have indeed located a critical point immediately below which the SU(5)–SU(5)/Z_5 system behaves smoothly, we perform hysteresis cycles on this system for large fixed negative β_A including both the strong- and weak-coupling regions. In figures 3(*a*) and (*b*) we take $\beta_A = -11.0$ and -14.0, respectively. Rather than the first-order transition of the Wilson theory, we see only signs of a cross-over reminiscent of the conventional SU(2) and SU(3) models.

Acknowledgments

We would like to thank the Science and Engineering Research Council of Great Britain for the award of research grants (Grants NG-0983.9 and NG-1068.2) to buy time on the CRAY-1S computer at Daresbury Laboratory where part of this calculation was carried out and the University of London Computer Centre for the granting of discretionary time on their CRAY-1S where the rest of this calculation was performed. The calculations of the present paper took 8.5 h of CRAY-1S time to perform. One of the authors (KJMM) would like to thank the Harry D Pierce Visiting Professorship Fund at the Technion for making his visit to the Technion, where this work was completed, possible. This research was also carried put in part under the auspices of the US Department of Energy under contract no DE-AC02-76CH00016.

References

Ardill R W B, Creutz M and Moriarty K J M 1983a *Phys. Rev.* D**27** 1956
Ardill R W B, Moriarty K J M and Creutz M 1983b *Comput. Phys. Commun.* **29** 97
Bachas C P and Dashen R F 1982 *Nucl. Phys.* B **210** [FS6] 583
Bhanot G 1982 *Phys. Lett.* **108B** 337
Bhanot G and Creutz M 1981 *Phys. Rev.* D **24** 3212
Bhanot G, Heller U M and Neuberger H 1982 *Phys. Lett.* **115B** 237
Bhanot G and Moriarty K J M 1983 *Phys. Lett.* **122B** 271
Bohr H and Moriarty K J M 1981 *Phys. Lett.* **104B** 217
Creutz M 1979 *Phys. Rev. Lett.* **43** 553
—— 1981 *Phys. Rev. Lett.* **46** 1441
Creutz M, Jacobs L and Rebbi C 1979 *Phys. Rev.* D **20** 1915

Creutz M and Moriarty K J M 1982a *Phys. Rev.* D **25** 1724
—— 1982b *Phys. Rev.* D **26** 2166
—— 1982c *Nucl. Phys.* B **210** [FS6] 50
Eguchi T and Kawai H 1982 *Phys. Rev. Lett.* **48** 1063
Greensite J and Lautrup B 1981 *Phys. Rev. Lett.* **47** 9
Halliday I G and Schwimmer A 1981a *Phys. Lett.* **101B** 327
—— 1981b *Phys. Lett.* **102B** 337
Metropolis N, Rosenbluth A W, Rosenbluth M N, Teller A H and Teller E 1953 *J. Chem. Phys.* **21** 1087
Moriarty K J M 1981 *Phys. Lett.* **106B** 130