RENORMALIZATION–GROUP STUDY OF FOUR-DIMENSIONAL SU(4) GAUGE THEORY

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Received 23 May 1983
Revised manuscript received 24 June 1983

Wilson loops for pure SU(4) gauge theory are calculated by Monte Carlo simulations on an 8^4 lattice. By comparing Wilson loops on different length scales, an ultraviolet stable attractive fixed point is located at vanishing bare coupling at which point asymptotic freedom is numerically verified.

In some recent papers, we studied pure SU(4) gauge theory with the Wilson action on a hypercubical lattice in four space–time dimensions and found a first-order phase transition at $\beta = 10.2$ [1], that this phase transition was not deconfining [2] and measured the asymptotic freedom scale parameter Λ_0 [2]. In the present paper, we wish to apply a renormalization-group scheme [3] for comparing the bare coupling constants at different cutoff values, to the gauge group SU(4). This procedure was previously successfully applied to SU(2) [3] and SU(3) [4]. In order to implement this renormalization-group procedure to SU(4), we measure all Wilson loops up to size 4×4 on an 8^4 lattice.

Although the gauge group of interest to the strong interactions is SU(3), there are interesting reasons to extend these renormalization group studies to SU(4). First, considerable work has gone into using the inverse of the group size as a possible expansion parameter. Going to a larger group should improve the convergence of this expansion and permit testing its details. Second, the phase transition structure of the pure Wilson theory changes qualitatively at SU(4), where a first order transition first appears. This transition is presumably an artifact of the formulation and should not have any effect on continuum results. It is of interest therefore to see if the presence of this transition significantly alters conclusions based on physical ratios of loops, as used in the previous studies with SU(2) and SU(3).

Since our calculational procedure is described in great detail in refs. [3,4], the description in the present paper is brief. We work on a hypercubical lattice in four euclidean space–time dimensions [5,6]. The link joining the nearest-neighbor lattice sites i and j is denoted by $\{i, j\}$ and on this link sits an $N \times N$ unitary-unimodular matrix U_{ij} of the gauge group SU(N) with

$$U_{ji} = (U_{ij})^{-1} .$$

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We define our partition function by

$$Z(\beta) = \int \left(\prod_{\{i,j\}} \mathrm{d}U_{ij} \right) \exp(-\beta S[U]) ,$$

where β is proportional to the inverse coupling constant squared, $\beta = 2N/g_0^2$, where g_0 is the bare coupling constant, and $\mathrm{d}U_{ij}$ is the normalized invariant Haar measure for SU(N). The action S is defined as a sum over all unoriented plaquettes □ such that

$$S[U] = \sum_{\square} S_{\square} = \sum_{\square} (1 - N^{-1} \operatorname{Re} \operatorname{tr} U_{\square}) .$$

Here $U_{\square}$ represents the product of link variables around the given plaquette. Periodic boundary conditions were used throughout our calculations and the lattice was equilibriated by the method of Metropolis et al. [7]. Our calculations were carried out on a vector processor, the CDC CYBER 205. Full details of the vectorization of our calculations is given in ref. [8]. From now on we specialize to SU(4).

We define our Wilson loops [9] by the expectation value

$$W(I, J) = \tfrac{1}{4}\langle \operatorname{Re} \operatorname{tr} U_C \rangle ,$$

where the rectangle C has length I and width J and U_C is the product of link variables around C. The logarithmic ratios $\chi(I, J)$ defined by

$$\chi(I, J) = -\ln \left(\frac{W(I, J) W(I-1, J-1)}{W(I, J-1) W(I-1, J)} \right),$$

are used to extract the string tension.

Following ref. [3], we introduce ratios of Wilson loops, which have the same perimeter and the same number of corners. This removes the associated ultraviolet divergences. These ratios are defined on length scales differing by a factor of two. Thus, we consider

$$F(\beta) = 1 - \frac{W(1,1) W(2,2)}{W(2,1) W(2,1)}, \qquad (1)$$

and

$$G(\beta) = 1 - \frac{W(2,2) W(4,4)}{W(4,2) W(4,2)}. \qquad (2)$$

The leading-order weak-coupling expansions are

$$F(\beta) = G(\beta) + O(\beta^{-2}) + O(a^2/r^2\beta)$$
$$= 8p_1/\beta + O(\beta^{-2}) + O(a^2/p^2\beta) , \qquad (3)$$

where $p_1 = 0.123\,899\,43$. The final term in eq. (3) represents finite cutoff corrections, with r being the scale of the observable under consideration; see ref. [3] for more detail. A renormalization-group fixed point β_F would occur when the observables are scale invariant, i.e. when

$$F(\beta_F) = G(\beta_F) . \qquad (4)$$

Eq. (7) of ref. [4] suggests that at weak coupling, with a suitable shift in the inverse coupling constant squared, the two functions $F(\beta)$ and $G(\beta)$ should be the same with

$$F(\beta) = G(\beta + (44/3\pi^2) \ln 2) + O(\beta^{-3}) . \qquad (5)$$

We performed our calculations by first carrying out 20 iterations through the 8^4 lattice with 16 Monte Carlo updates per link. The Wilson loops were then calculated over the next 240 iterations through the lattice. In fig. 1 we show some ordered phase starting runs for the Wilson loops. Fig. 1a shows such a run at $\beta = 10.4$ which is near the critical inverse coupling constant squared. From this diagram we can see that $W(3,3)$ and $W(4,4)$ need 200 iterations to equilibriate. Fig. 1b shows a run at $\beta = 14.0$ which is well within the weak-coupling region. This diagram shows that the space–time lattice equilibriates for all Wilson loops up to size 4 × 4 after about 50 iterations through the lattice. Thus, for all our calculations we averaged over

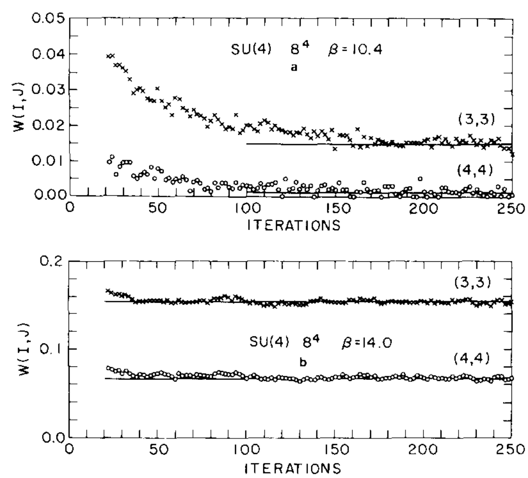

Fig. 1. The evolution of the Wilson loops $W(I, J)$ for pure SU(4) gauge theory of an 8^4 lattice as a function of the number of iterations through the lattice for ordered phase starting lattices for $\beta = 10.4$ (a) and 14.0 (b).

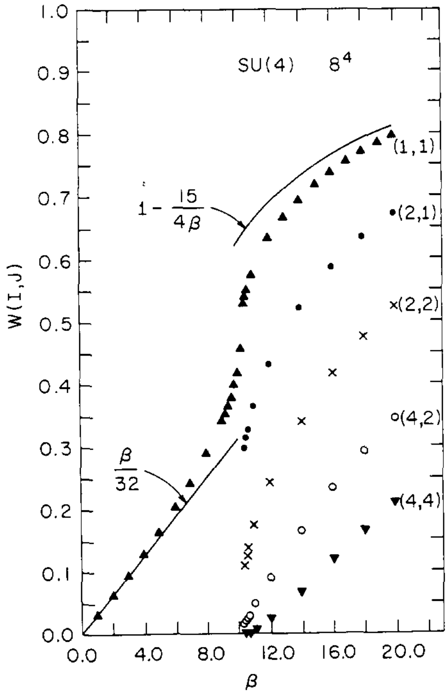

Fig. 2. The Wilson loops $W(I, J)$ for pure SU(4) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β. The full upward triangles represent (I, J) = (1,1), the open circles represent (2,1), the crosses represent (2,2), the full downward triangles represent (4,2) and the full squares represent (4,4). The curves represent the leading-order strong- and weak-coupling expansion of eqs. (1) and (2) (of ref. [4]), respectively.

the last 60 iterations through the lattice where, in order to reduce the correlations between events, only every second iteration was included in the average. Thus, 30 lattice configurations were used in the average. Ordered starting lattices were used throughout our calculations. One update per link took 300 μs on the CDC CYBER 205.

In fig. 2 we show the Wilson loops 1 × 1, 2 × 1, 2 × 2, 4 × 2 and 4 × 4 as a function of β. The leading-order strong- and weak-coupling expansions of eqs. (1) and (2) (of ref. [4]), respectively are also shown.

In order to make the present paper more useful to the reader and to give the reader some idea of the

Table 1
Data for the Wilson loops

	β						
	10.4	10.5	10.6	11.0	12.0	14.0	16.0
⟨W(1,1)⟩	0.5255 ±0.0076	0.5390 ±0.0053	0.5484 ±0.0049	0.5783 ±0.0029	0.6297 ±0.0016	0.6960 ±0.0010	0.7406 ±0.0009
⟨W(2,1)⟩	0.2985 ±0.0106	0.3156 ±0.0076	0.3279 ±0.0071	0.3662 ±0.0042	0.4331 ±0.0025	0.5235 ±0.0017	0.5875 ±0.0014
⟨W(2,2)⟩	0.1135 ±0.0114	0.1288 ±0.0084	0.1407 ±0.0080	0.1761 ±0.0052	0.2439 ±0.0035	0.3418 ±0.0026	0.4181 ±0.0024
⟨W(3,1)⟩	0.1726 ±0.0106	0.1884 ±0.0076	0.2002 ±0.0073	0.2366 ±0.0045	0.3040 ±0.0029	0.4001 ±0.0020	0.4725 ±0.0019
⟨W(3,2)⟩	0.0467 ±0.0090	0.0570 ± 0.0067	0.0656 ±0.0067	0.0918 ±0.0048	0.1475 ±0.0035	0.2355 ±0.0029	0.3111 ±0.0028
⟨W(3,3)⟩	0.0151 ±0.0060	0.0203 ±0.0049	0.0259 ±0.0049	0.0417 ±0.0040	0.0818 ±0.0035	0.1530 ±0.0030	0.2224 ±0.0033
⟨W(4,1)⟩	0.1002 ±0.0091	0.1132 ±0.0066	0.1228 ±0.0063	0.1535 ±0.0041	0.2147 ±0.0028	0.3071 ±0.0022	0.3814 ±0.0023
⟨W(4.2)⟩	0.0195 ±0.0062	0.0262 ±0.0049	0.0314 ±0.0047	0.0490 ±0.0038	0.0912 ±0.0031	0.1649 ±0.0030	0.2347 ±0.0031
⟨W(4,3)⟩	0.0049 ±0.0037	0.0077 ±0.0032	0.0110 ±0.0031	0.0199 ±0.0030	0.0474 ±0.0028	0.1026 ±0.0029	0.1632 ±0.0036
⟨W(4,4)⟩	0.0012 ±0.0023	0.0026 ±0.0022	0.0041 ±0.0024	0.0092 ±0.0022	0.0264 ±0.0024	0.0670 ±0.0028	0.1184 ±0.0039

statistical errors in our SU(4) Wilson loop data, we present this data in tabular form in table 1. The quoted errors are the standard deviation measurements on the data. As we can see the errors are too small, relative to the data points, to be plotted in fig. 2. Of course, we have no way of estimating our systematic errors. Our results for Wilson loops up to size 2 × 2 agree with previous measurements [1] made on a 4^4 lattice which suggests that finite-size effects are at a minimum. A number of other statistical tests, e.g., changing the sequence of random numbers used for sampling, taking averages over different sequences of equilibriated gauge field configurations and using a different number of iterations before deciding to take our averages, all led to similar results indicating the numerical stability of our results.

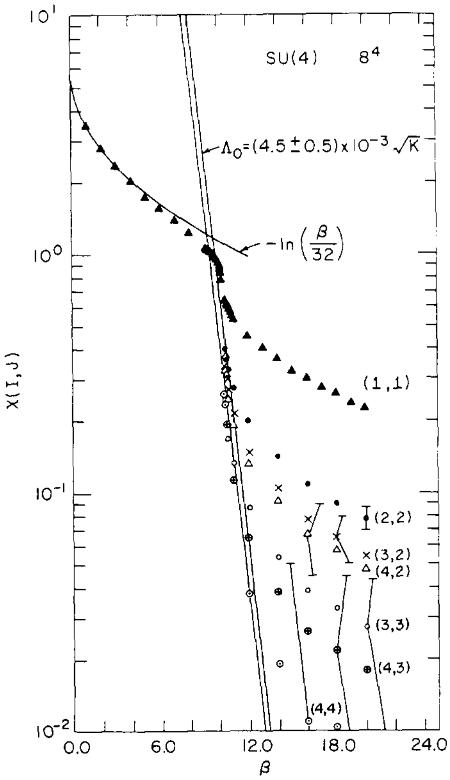

Fig. 3. The string tension $\chi(I, J)$ for pure SU(4) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β. The full upward triangles represent (I, J) = (1, 1), the full circles represent (2,2), the crosses represent (3,2), the open triangles represent (4,2), the open circles represent (3,3), the cricles with crosses represent (4,3) and the circles with dots represent (4,4). The leading-order strong-coupling expansion of eq. (3) of ref. [4] is also shown.

Fig. 3 shows the logarithmic rations $\chi(I, J)$ for (I, J) = (1,1), (2,2), (3,2), (4,2), (3,3), (4,3) and (4,4) as a function of β. The band in fig. 3 corresponds to the functional form of eq. (6) of ref. [4] with the asymptotic freedom scale parameter taken from ref. [2]

$$\Lambda_0 = (4.5 \pm 0.5) \times 10^{-3}\sqrt{K}\,, \tag{6}$$

where K is the string tension. Note that the Monte Carlo data appear somewhat steeper than the scaling prediction. This suggests that the nearby transition is delaying the onset of scaling and that $\Lambda/\sqrt{K}$ may be somewhat larger than the earlier statement in eq. (6). The errors in fig. 3 are the maximum standard deviation errors measured in the averaging procedure. We feel that as a result of these errors, the results derived from the measurements of the larger Wilson loops have statistical significance.

In fig. 4 the quantities F and G are shown as functions of the inverse coupling constant squared β. Also plotted are the leading-order strong-coupling expansions of eqs. (10) and (11) of ref. [4] and the weak-coupling expansion of eq. (3). From fig. 4 we see that $G(\beta) > F(\beta)$ for all β with the only fixed point, following eq. (4), evident at $\beta = \infty$.

$F(\beta)$ and $G(\beta + (44/3\pi^2) \ln 2)$, are shown in fig. 5 as a function of the inverse coupling constant squared β. Agreement with the asymptotic freedom prediction of eq. (5) is achieved. In figs. 4 and 5 the errors shown are the maximum standard deviation measurements made during the averaging process.

As in ref. [3], we can define an effective renormal-

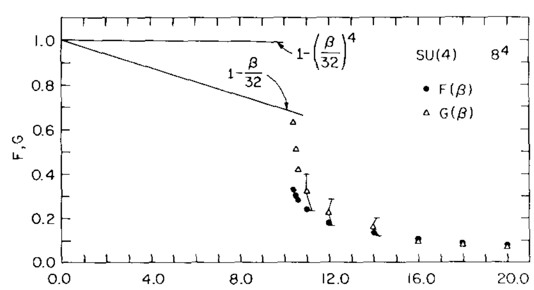

Fig. 4. The quantities $F(\beta)$ and $G(\beta)$ for pure SU(4) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β. The curves represent the leading-order strong- and weak-coupling expansions of eqs. (10) and (11) of ref. [4] and eq. (3).

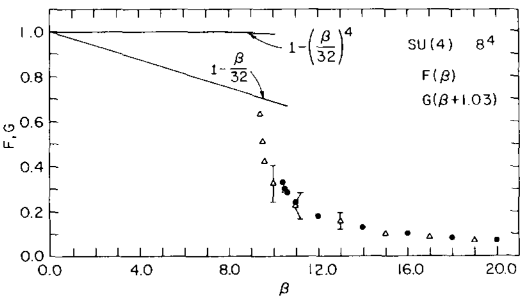

Fig. 5. The quantities $F(\beta)$ and $G(\beta + 1.03)$ for pure SU(4) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β.

ized coupling at twice the lattice spacing

$$g_R^2(2a) = F(\beta)/p_1 \ .$$

In fig. 6 we plot the inverse of this coupling versus β. Also shown in the leading-order strong-coupling expansion. The curve in the weak-coupling region has a slope of 1/8. The good agreement of this slope with the Monte Carlo data indicates that $(a/r)^2$ corrections in eq. (5) are small. The intercept in fig. 6 gives

$$2\gamma_0 \ln(2\Lambda_R/\Lambda_0) \simeq 0.9 \ ,$$

or

$$\Lambda_R = 63.5 \ \Lambda_0 \ .$$

A closely related number, using the interquark potential to define a renormalized coupling, has been calculated perturbatively by Billoire [10] to be 52.2. From eq. (6) we obtain

$$\Lambda_R = 0.29\sqrt{K} \ .$$

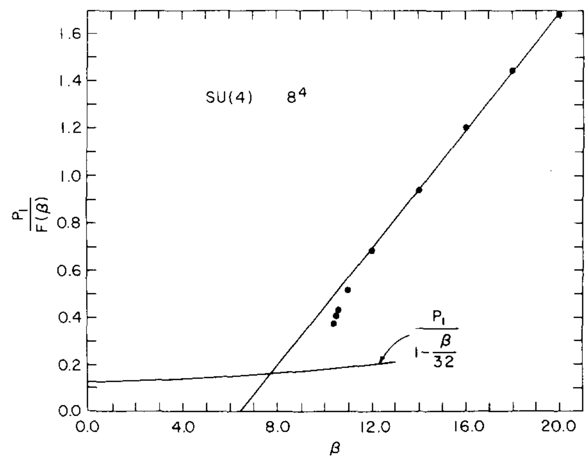

Fig. 6. The quantity $p_1/F(\beta)$ for pure SU(4) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β. The curves represent the leading-order strong-coupling expansion of eq. (3) and the straight-line of eq. (7) of ref. [4].

We would like to thank the Control Data Corporation for the award of a research grant (Grant No. 82CSU13) to buy time on the CDC CYBER 205 at the Institute for Computational Studies at Colorado State University. The calculations of the present paper took 9.45 hours of CDC CYBER 205 time to perform. One of the authors (K.J.M.M.) wishes to thank the DESY Directorate for the award of a Visiting Fellowship to visit DESY where this work was completed. This research was also carried out in part under the auspices of the US Department of Energy under contract No. DE-AC02-76CH00016.

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