BROOKHAVEN NATIONAL LABORATORY

June 1990 BNL – 44743

Abelian Sandpiles

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ABSTRACT

A class of models for self organized critical phenomena possesses an isomorphism between the recursive states under addition and the abelian operator algebra on them. Several exact results follow, including the existence of a unique identity state, which when added to any configuration C in the recursive set relaxes back to that configuration. In this relaxation process, the number of topplings at any lattice site is independent of the configuration C.

This manuscript has been authored under contract number DE-AC02-76CH00016 with the U.S. Department of Energy.

I. Introduction

The concept of "self organized criticality" describes the tendency of strongly dissipative systems to show relaxation behavior involving a wide range of length and time scales.[1] The phenomenon is expected to appear widely; indeed, it has been looked for in such diverse areas as earthquake structure[2] and economics.[3] The idea provides an alternative view of complex behavior in systems with many degrees of freedom. It complements the concept of "chaos," wherein simple systems with a small number of degrees of freedom can display quite complex behavior.

The prototypical example of self organized criticality is a sandpile. Adding sand slowly to a heap of sand will result in the slopes increasing to a critical value, where an additional grain will give rise to an unpredictable behavior. If the slope were too steep, one would obtain a large avalanche and a collapse to a more stable configuration, while if it were less steep the new sand will just accumulate to make the pile steeper. At the critical slope the distribution of avalanches has no overall scale, but rather show a power law behavior. This has been extensively discussed by several authors. [4–8]

Ref. 1 presented a simple mathematical system to demonstrate this behavior. This is a cellular automaton model formulated on a regular lattice, and uses an integer variable on each site to represent the local sandpile slope. When the slope exceeds a critical value, an avalanche ensues, with sand spilling onto neighboring sites, changing their local slopes. Studies of this model gave evidence of a self organized critical state where the distribution of avalanches was indeed a power law.

This particular model for self organized criticality was recently shown to have rather remarkable properties.[9] In particular, the system is described in termsof an Abelian group. If two grains of sand are added to the system in arbitrary locations, the resulting final state is independent of the orders of addition and the intermediate relaxation steps. This result enables an exact calculation of the number of states important in the self organized critical state and determination of the average number of tumblings occurring at a site i given a grain of sand has been added at site j. While this does not determine all critical properties of interest, it suggests that further exact results may be found for this special model.

The main new results of this paper include the demonstration of an isomorphism between the recursive (defined in Ref. 9 and below) subset of the states under this addition and the operator algebra generated by sand addition. I show how to construct the unique non-trivial state corresponding to the identity under this isomorphism. When this state is added to any other state C in the recursive set, the resulting state always relaxes back to C. I also show that when any recursive state is added to the identity and the system relaxed, the number of topplings at any site depends only on the site position and not on the state being added.

The next section defines the model and several quantities useful in its description. Section III is primarily a review of Ref. 9, and shows how operations of adding sand form a discrete Abelian group. Section IV defines addition between states and shows the isomorphism alluded to above. This section contains a simple algorithm for explicitly constructing the identity state. Section V presents some results on the numbers of topplings coming from the addition of various states. Section VI contains a few minor conclusions.

II. Defining the model

For definiteness I consider a finite two dimensional rectangular lattice of size $N_x \times N_y$. The total number of sites is $N = N_x N_y$. On each site i of this lattice is an integer value z_i representing the local slope. I will also loosely refer to z_i as the amount of sand on site i, although the analogy with real sandpiles is perhaps better if it is regarded as a slope.

Definition 1. The set $\mathcal{G}$ of general states is the set of configurations C specified by an arbitrary integer z_i on each lattice site i.

This system is to be updated with a simple cellular automaton rule. The dynamics involves a threshold value for the slope z_T. Wherever the local slope z_i is larger or equal to this threshold value, the corresponding site is regarded as unstable. One updating step consists of first determining the set of all unstable sites. Then simultaneously 4 is subtracted from the unstable slopes and 1 is added to each unstable site's neighbors. This action on an unstable site is referred to as a "toppling."

Ref. 9 introduced a useful matrix Δ_{ij} with integer elements representing the change in the slope z at site i resulting from a toppling at site j. The indices in this matrix run over the sites of the lattice. For the simple two dimensional rectangular geometry considered here, I have

Definition 2. The toppling matrix is given as

$$\Delta_{ij} = \begin{cases} +4 & i = j \\ -1 & i \text{ and } j \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Under a toppling at site j, the slope at site i becomes $z_i - \Delta_{ij}$.

There is nothing particularly special about the lattice geometry; indeed, the discussion easily generalizes to other lattices and dimensions. On a Cayley tree the model can be solved exactly.[10] The discussion in Ref. 9 is quitegeneral, requiring only that under a toppling at a single site i, (1) that site has its slope decrease ($\Delta_{ii} > 0$), (2) the slope at any other site is either increased or unchanged ($\Delta_{ij} \geq 0,\ i \neq j$), (3) the total amount of sand in the system does not increase ($\sum_i \Delta_{ij} \geq 0$), and (4) each site be connected through topplings to some location where sand can be lost, such as at a boundary ($\exists j$ such that $\sum_i \Delta_{ij} > 0$). All results discussed here apply to this more general model.

For the specific case in Eq. (1), whenever a site i away from the lattice edge undergoes a toppling, the sum of slopes over all sites is conserved. Only at the lattice boundaries can sand be lost. Thus the details of this model depend crucially on the boundary conditions, which I take to be open. A a toppling on the middle of an edge looses one grain of sand and at a corner looses two.

The actual value of the threshold z_T is unimportant to the dynamics. This can be changed by merely adding a constant to all the z_i. Thus without loss of generality I consider

$$z_T = 4 \tag{2}$$

With this convention, if all z_i are initially non-negative, they will remain so after any addition of sand or number of updating steps. Thus it is also convenient to define the set of non-negative states $\mathcal{P}$:

Definition 3. $\mathcal{P} = \{C \in \mathcal{G} | z_i \geq 0\ \forall i\}$.

If the updating procedure finds no unstable sites, then the state C is said to be stable, and undergoes no changes. I thus am led to define the stable subset of $\mathcal{P}$:

Definition 4. $\mathcal{S} = \{C \in \mathcal{P} |\ z_i \leq 4\ \forall i\ \}$. is referred to as the set of stable states.

If a state has any z_i larger or equal to $z_T = 4$, it is called unstable.

One important state in $\mathcal{S}$ is the minimally stable state,[11] which has all slopes just below the threshold; that is

Definition 5. The minimally stable state C^* is that stable state with $z_i = z_T = 3\ \forall i$.

By construction, any addition of sand to C^* will give an unstable state.

It is now convenient to introduce some notation for various operators which can be applied to these states. First I define the operator α_i to of add one grain of sand to the slope at site i:

Definition 6. Given a state $C \in \mathcal{G}$ and its attendant slopes z, the state $\alpha_i C \in \mathcal{G}$ is the state with slopes z' where

$$z'_j = \begin{cases} z_i + 1 & \text{if i=j} \\ z_j & \text{otherwise} \end{cases}$$

This procedure can be applied to any state, stable or not. Applied to a stable state, it might or might not make that state unstable, depending on the previous value of z_i.

The next useful operator t_i represents a toppling at site i:

Definition 7. Given a state C and its attendant slopes z_j, the state $t_i C$ is the state with slopes z'_j where

$$z'_j = z_j - \Delta_{ji}$$

The operator which updates the lattice one time step is now simply the product of t_i over all sites where the slope is unstable:

Definition 8. The updating operator U applied to state C with slopes z gives the state

$$UC = \prod_i t_i^{p_i} C$$

where

$$p_i = \begin{cases} 1 & \text{if } z_i \geq 4 \\ 0 & \text{otherwise} \end{cases}$$

Using U repeatedly, we can define the relaxation operator R. Applied to any state C, this corresponds to repeatedly applying U until no more z_i change. Both R and U have no effect on any stable state.

At this point is not entirely clear that the operator R exists; that is, it might be that the updating procedure enters a non-trivial cycle. We now prove that this is impossible.

Theorem 1. *On any state $C \in \mathcal{P}$ the updating procedure will always converge to a stable state RC.*

Proof. We need only show that the updating rule cannot give rise to a nontrivial cycle of unstable states. First note that a toppling in the interior of the lattice does not change the total amount of sand $\sum_i z_i$. A tumbling on the boundary, however, decreases this sum due to sand falling off the edge. Thus this sum is non-increasing and will decrease whenever there is a toppling on the boundary. Any cycle, therefore cannot have tumblings on the boundary. Next note that any tumblings one site away from the boundary will increase the sand contained on the boundary sites. Thus the sand in the boundary will monotonically increase if there is any tumbling one site away. This cannot happen in a cycle, thus there can be no tumblings on sites one step away from the edges. This argument inductively repeats to arbitrary distances from the boundary; thus, the cycle is trivial.

Note that for a general geometry this result requires that every site be eventually connected to an edge where sand can be lost. With periodic boundaries no sand would be lost and thus cycles are expected. ∎

Having shown that the relaxation operator exists, I consider the action of adding a grain of sand followed by relaxation

Definition 9. On any stable state C the operator a_i gives the state $a_i C = R\alpha_i C$.

Ref. 9 has introduced the concept of "recursive states." The set $\mathcal{R}$ of such states includes those stable states any one of which can be obtained from any other by some addition of sand followed by relaxation. As the minimally stable state C^* can be obtained from any other state by adding just enough sand to each site to make z_i equal to three, it is in $\mathcal{R}$. Thus a convenient definition of the recursive set is

Definition 10. A state C is in the recursive set $\mathcal{R}$ if and only if it can be obtained from the state C^* by acting with some product of the operators a_i.

It is easily shown that the difference between $\mathcal{S}$ and $\mathcal{R}$ is not empty. In particular, a recursive state can never have two adjacent slopes being both zero. A scheme for determining if a state is recursive was given in Ref. 9; I give another method in section IV. Ref. 9 also shows that the self organized critical ensemble, reached under random addition of sand to the system, has equal probability for each state in the recursive set.

III An Abelian group

This section is primarily a review of Ref. 9. The crucial results are that the operators a_i on stable states all commute, and that they generate an Abelian group when restricted to recursive states. These remarkable properties are at the root of all the remaining results in this paper. I begin with

Theorem 2. *The operators a commute; that is:*

$$a_i a_j C = a_j a_i C \;\forall\; i, j, C \in \mathcal{S}. \tag{3}$$

Proof. Given i, j, and C, there exists a set of topplings t_{l_k}, $k = 1, \dots n1, \dots n$ such that

$$a_i a_j C = \left(\prod_{k=n}^{n_1} t_{l_k} \right) \alpha_i \left(\prod_{k=n_1+1}^{n} t_{l_k} \right) \alpha_j C \tag{4}$$

here the specific numbers of topplings n_1 and n depend on i, j, and C. Acting on general states, the operators t and α all commute because they merely linearly translate the slopes z. A general such commutation might force a state outside $\mathcal{P}$, but will remain in $\mathcal{G}$. Thus we have

$$a_i a_j C = \left(\prod_{k=1}^{n} t_{l_k} \right) \alpha_i \alpha_j C \tag{5}$$

Note that in moving α_i to the right, the slopes encountered by the toppling operators t_{l_k} all increase. Thus if these intermediate slopes were all positive they remain so.

Now I do a rearrangement of the product of topplings. As t_i can reduce the slope only on site i, there must exist in this product of topplings at least one factor of t_i for every site with $z_i \geq 4$. I move the rightmost of each of these factors to the right of the product and thus build up a factor of the updating operator.

$$a_i a_j C = \left(\prod t_{l_k} \right) U \alpha_i \alpha_j C \tag{6}$$

where the product now is only over the remaining toppling factors. Because t_i increases slopes at all sites $j \neq i$ and site i itself starts with a slope larger than 4, we continue to have the property that if all intermediate slopes were positive, they remain so after this rearrangement.

This procedure can be repeated to remove more factors of U until we have a state in the stable set. At this point we must have exhausted all factors of t_{l_k}. Thus we have

$$a_i a_j C = R\alpha_i \alpha_j C = R\alpha_j \alpha_i C = a_j a_i C \tag{7}$$

∎

The construction in this proof also gives an interesting result on the number of topplings occurring when the sand is added:

Corollary 1. *Given C in the stable set, the numbers of tumblings at any site k occurring in the operations $a_i a_j C$ and $a_j a_i C$ are the same.*

Of course, if a site k tumbles, it might be "caused" by either addition; the orders of the tumblings in the two cases may or may not be altered.

Note that because the ordering of tumblings is unimportant, repeated addition of sand, even to only one site, will eventually produce a state which relaxes into the recursive set. With sufficient sand available, selectively ordered tumblings can spread it over the lattice to make all slopes supercritical.

I now restrict myself to the recursive set. In this set we have the remarkable result that the operator a_i has a unique inverse:

Theorem 3. *Given a_i, then $\forall C \in \mathcal{R}$ there exists a unique $(a_i^{-1}C) \in \mathcal{R}$ such that $a_i(a_i^{-1}C) = C$.*

This theorem implies that the operators a_i when acting on the recursive set form an abelian group.

Proof. First I find one recursive state such that a_i on it gives C and then I will show that this state is unique. I begin by adding a grain of sand to the minimally stable state C^* and allowing it to relax. This generates a new recursive state $a_i C^*$. Now add sand to this state selectively to regenerate the state C^*; that is, construct

$$P_1 = \prod_i a_i^{3-z_i} \tag{8}$$

where z_i are the slopes in the state $a_i C^*$. By construction I have

$$P_1 a_i C^* = C^*. \tag{9}$$

Now by assumption C is itself a recursive state. This means there is a product P_2 of the a_i such that

$$C = P_2 C^*. \tag{10}$$

Combining things and using the Abelian nature of the operators, I have

$$C = P_2 P_1 a_i C^* = a_i P_2 P_1 C^* \tag{11}$$

Thus $P_2 P_1 C^*$ is a recursive state on which a_i gives C.

I must still show that this state is unique. To do this consider repeating the above process to find a sequence of states $C_n \in \mathcal{R}$ satisfying

$$(a_i)^n C_n = C \tag{12}$$

Because on our finite system the total number of stable states is finite, this sequence must eventually enter a loop. To run around this loop in the other direction, mere repeatedly operate with a_i on the states. But such an operation eventually returns to the original configuration C, which therefore must lie in the loop. Calling the length of the loop m, I have $a_i^m C = C$. I then uniquely define $a_i^{-1} C = a_i^{m-1} C$. ∎

I am now ready to count the number of recursive states.

Theorem 4. *The number of recursive states equals the absolute value of the determinant of the toppling matrix* Δ.

Proof. As all recursive states can be obtained by adding sand to C^*, we can write any state $C \in \mathcal{R}$ in the form

$$C = \left(\prod_{i=1}^{N} a_i^{n_i} \right) C^* \tag{13}$$

Here the integers n_i represent the amount of sand to be added at the respective sites. However, in general there are several different ways to reach any given state. In particular, adding four grains to any one site will force a toppling and is equivalent to adding a single grain to each of its neighbors. As an operator statement, I have

$$a_i^4 = \prod_{j \in \mathcal{N}(i)} a_j \tag{14}$$

where $\mathcal{N}(i)$ denotes the set of nearest neighbors to site I. This is true on any stable state, but by restricting operations to the recursive set, where inverses exist, I can rewrite this equation

$$\prod_j a_j^{\Delta_{ij}} = E \tag{15}$$

where E denotes the identity operator. Eq. (15) is valid for any i. This relation allows us to change the powers appearing in Eq. (13). If we now label states by the vector of powers n_i, $i = 1, \ldots N$, we see that two such states are equivalent if the difference of these vectors is of the form $\sum_j \beta_j \Delta_{ji}$ where the coefficients β_j are integers. These are the only constraints; if two states cannot be related by toppling, they are independent. Thus any vector n_i can be translated to lie in an N dimensional hyper-parallelopiped N whose base edges are the vectors Δ_{ji}, $j = 1 \ldots N$ extending from the origin. The vertices of this object have integer coordinates and its volume is the number of integer coordinate points inside it. This volume is just the absolute value of the determinant of Δ. ∎

The structure of finite Abelian groups is well understood.[12] Many results,however, depend in detail on the prime factors of the order of the group. Here the divisors of $|\Delta|$ are sensitive functions of the lattice dimensions, about which I know little in general. Nevertheless, since any element of a group raised to the order of the group gives the identity, it follows that on the recursive set

$$a_i^{|\Delta|} = E. \tag{16}$$

Depending on the detailed lattice dimensions, it may be that a lower power of a_i is the identity, but this power must always be a divisor of $|\Delta|$. For some explicit examples on small systems, a 2×2 lattice has $|\Delta| = 192$ but $a_i^{24} = E$ for any i. On a 2×7 lattice, however, all 59,817,135 recursive states can be reached by adding sand to only one corner site. In all cases, however, Eq. (16) implies that

$$a_i^{-1} = a_i^{|\Delta|-1}. \tag{17}$$

Finally note that for large volume this determinant can be easily found by Fourier transform. In particular, whereas there are 4^N stable states, there are only

$$\exp\left\{ N \int_{(-\pi,-\pi)}^{(\pi,\pi)} \frac{d^2q}{(2\pi)^2} \log\left(4 - 2\cos q_x - 2\cos q_y\right) \right\} = (3.2102\ldots)^N$$

recursive states.

Eq. (15) also implies that some of the a_i can be eliminated in terms of others. In particular, addition of any sand to any lattice row can be replaced by additions to previous rows. Iterating this gives the amusing result that any recursive state can be reached from any other by only adding sand to a single edge of the system. In some cases, such as the 2 by 7 lattice mentioned above, a further reduction is possible, but this depends on the specific dimensions of the lattice.

IV. An isomorphism

I now pursue the consequences of combining states together.

Definition 11. Given stable configurations C and C' with the respective slopes z_i and z_i'. The state $C \oplus C'$ is defined to be the state obtained by relaxing the configuration with initial slopes $z_i + z_i'$.

Clearly if either C or C' are recursive states, then $C \oplus C'$ is as well. If I now restrict consideration to recursive states alone, I have

Theorem 5. *Under the operation $\oplus$ the states in $\mathcal{R}$ form an abelian group isomorphic to the algebra of the a_i operating on $\mathcal{R}$.*

Proof. Addition of a state C with slopes z_i is equivalent to operating with a product of a_i raised to the corresponding slopes. That is, given recursive states B and C

$$B \oplus C \leftrightarrow \left(\prod_i a_i^{z_i}\right) B \tag{18}$$

The operation $\oplus$ is associative and abelian because the operators a_i are.

Since on recursive states a_i^{-1} exists, there is an inverse to this addition of state C. For an explicit expression, consider the analog of Eq. (17)

$$-C = (|\Delta| - 1) \otimes C. \tag{19}$$

Here $n \otimes C$ means adding n copies of C with $\oplus$. The state $-C$ has the property that for any state B

$$B \oplus C \oplus (-C) = B \tag{20}$$

The state $I = C \oplus (-C)$ represents the identity and has the property $I \oplus B = B$, $\forall B \in \mathcal{R}$. This identity is unique.

Finally, the state isomorphic to the operator a_i is simply $a_i I$. ∎

The identity state provides a simple way to determine if a state is in the recursive state:

Corollary 2. *A stable state $C \in \mathcal{S}$ is in $\mathcal{R}$ if and only if $C \oplus I = I$.*

Proof. By construction, a recursive state has this property. On the other hand, since I is recursive, so is $I \oplus C$ ∎

I now give a simple algorithm for constructing the identity state. In principle one could take any recursive state, say C^*, and repeatedly add it to itself to use $|\Delta| \otimes C = I$. However, on any but the smallest lattices, $|\Delta|$ is a very large integer and this procedure would be computationally inconvenient. A simpler scheme starts by multiplying Eq. (15) over all sites. As an operator, this gives

$$\prod_j a_j^{\sum_i \Delta_{ij}} = E. \tag{21}$$

On our rectangular geometry, factors of a_i on interior sites will cancel in this product. The net effect reduces to adding a single grain of sand at each site on straight edges and two on the corners. Applying this operator to the empty state with all $z_i = 0$ gives a state I_0, generally outside the recursive set, with the property that when added to any recursive state C, it relaxes back to C. I now consider doubling this state and allowing the combination to relax, thus obtaining $I_1 = I_0 \oplus I_0$. This similarly does not alter a recursive state when added to it. This is then repeated, constructing $I_n = I_{n-1} \oplus I_{n-1}$. In a finite number of steps this procedure converges to $I_n = I_{n-1} = I \in \mathcal{R}$. In Figure 1 I show an example of the identity state on a lattice of size 288 by 188. Figure 2 shows an intermediate unstable state in the above construction of this state. Indeed, it is a visual feast to watch this creation of the identity.

V. Topplings

I now turn to some results on the number of topplings arising when two states are added together. For this purpose I give

Definition 12. Let $f_i(C,C') = f_i(C',C)$ denote the number of tumblings occurring at site i during the relaxation process in forming $C \oplus C'$.

Now consider adding three stable states C_1, C_2, and C_3. Because of Corollary 1, the order of addition does not matter to the total number of tumblings on any site, and we have the result

$$f_i\left(C_1, C_2\right) + f_i\left(C_1 \oplus C_2, C_3\right) = f_i\left(C_1, C_2 \oplus C_3\right) + f_i\left(C_2, C_3\right) \tag{22}$$

Now consider the tumblings occurring when the identity is added to a recursive state. Of course the total amount of sand lost in this relaxation must equal what is contained in the identity, but actually a much stronger result is possible.

Theorem 6. *For any given site i, the number $f_i(I,C)$ is independent of $C \in \mathcal{R}$.*

Proof. Inserting $C_2 = I$ in Eq. (22) and using $C \oplus I = C$ for recursive states gives the result. ▮

Let me now consider topplings generated by adding a single grain of sand to a configuration:

Definition 13. Consider a stable state C. Let $T_{ij}(C)$ denote the number of topplings occurring at site i during the relaxation of the state $a_j C$.

As recursive states play such an important role in the analysis of this model, one might wonder if $T_{ij}(C)$ can be obtained from the addition of recursive states alone. This is indeed possible if one uses as well the numbers of tumblings coming from adding sand to the identity state. Consider Eq. (22) with C_1 containing only a single grain of sand, $C_2 = I$, and C_3 being a general recursive state. Using also the result of Theorem 6, this gives the desired relation

$$T_{ij}\left(C\right) = T_{ij}\left(I\right) + f_i\left(a_j I, C\right) - f_i\left(I, I\right). \tag{23}$$

Thus for a recursive state we can obtain $T_{ij}(C)$ from properties of the identity and from additions of recursive states alone.

I end this section with a rederivation of the result of Ref. 9 for the average of $T_{ij}(C)$ over the recursive set. This average corresponds to an expectation in the self organized critical ensemble. Letting $z_i(C)$ denote the slope at site i for configuration C, the definition of the toppling matrix implies

$$z_i\,(a_j C) = z_i\,(C) - \Delta_{ik} T_{kj}\,(C) + \delta_{ij}. \tag{24}$$

Solving for T, I obtain

$$T_{ij}\,(C) = \left(\Delta^{-1}\right)_{ik} \left(z_k\,(C) - z_k\,(a_j C) + \delta_{kj}\right). \tag{25}$$

Averaging over recursive states, I have $\langle z_k(C)\rangle_{C\in\mathcal{R}} = \langle z_k(a_j C)\rangle_{C\in\mathcal{R}}$, implying

$$\langle T_{ij}\,(C)\rangle_{C\in\mathcal{R}} = \left(\Delta^{-1}\right)_{ij}. \tag{26}$$

VI. Conclusions

I have discussed some rather remarkable properties of a class of simple cellular automaton models for self organized criticality. In particular, these systems are characterized by a large Abelian group. Such special properties raise the question of whether the models may actually be solvable, in the sense that the exponents can be exactly found. On the other hand, they also raise the question of whether any critical behavior displayed by these models is generic.

While the average number of tumblings at any site from sand addition at another has been exactly determined, the fluctuations around this number have not. It is the latter that should be large in the critical state and are important to its properties. To determine the fluctuations would involve knowing correlations between the slopes before and after adding sand to the system. Such correlations may be an interesting topic for further study.

ACKNOWLEDGEMENTS

I am grateful to P. Bak and D. Dhar for discussions on the model, E. Meyers and D. Petcher for helping play with the system on the Connection Machine, and T. Bhattacharya for help in converting image formats for the figures. The figures were obtained on a standard Commodore Amiga.

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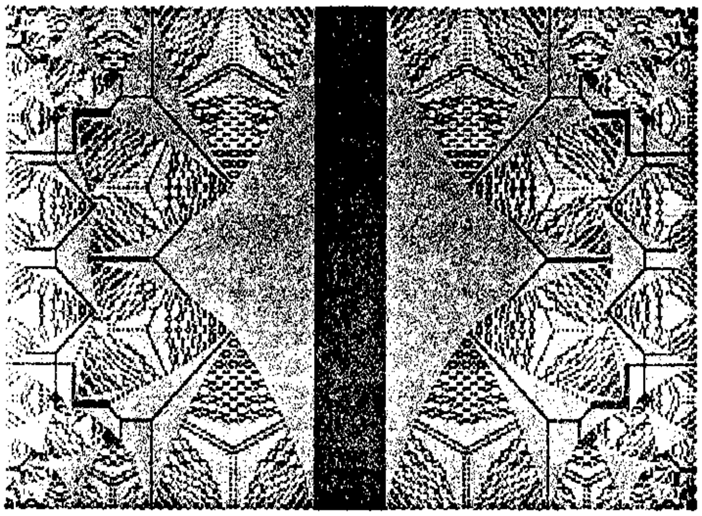

Figure 1. The identity state on a 288 by 188 lattice. Slopes from 0 to 3 are color coded as white, black, red, and green, respectively.

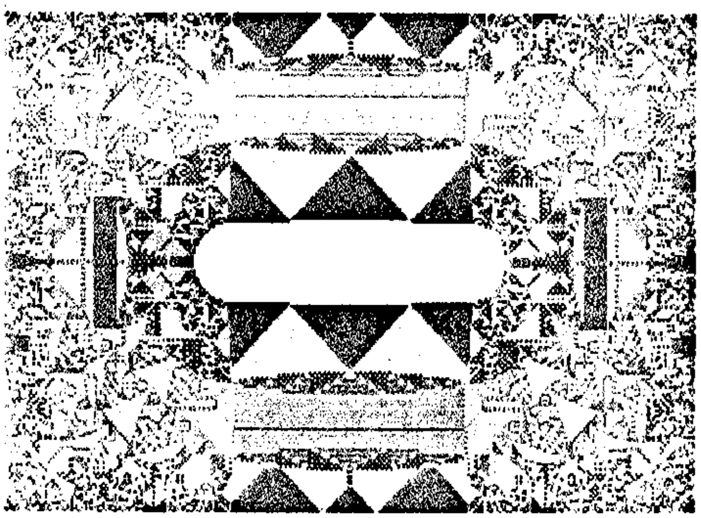

Figure 2. An intermediate active state in the generation of the identity state of fig. 1. Note the hole in the center where sand has not yet reached. Slopes from 0 to 7 are color coded as white, black, red, green, yellow, blue, magenta, and cyan, respectively.