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ABELIAN SANDPILES

Michael CREUTZ
Physics Department
Brookhaven National Laboratory
Upton, NY 11973
(creutz@wind.phy.bnl.gov)

A class of models for self organized criticality can be described in terms of a large Abelian group. Several exact results follow, including the existence of a unique non-trivial configuration representing the identity element.

In this talk I descibe some rather elegant mathematical properties of a simple cellular automaton model for self organized criticality. I will discuss how a subset of states of this model form an Abelian group. Then I will show how to construct the non-trivial state which represents the identity for this group. [1] The essence of this talk is a digression from the topic of this conference, but I will end with a brief mention of a tenuous connection with Gribov copies.

The model was first presented to study self organized criticality. [2] While this talk is not directly on that subject, let me begin by briefly summarizing this concept. Ref. 2 argued that strongly dissipative systems can drive themselves to a critical state. Unlike conventional critical phenomena, this should occur without any tuning of parameters to a critical value.

The prototypical example of this phenomenon is a sandpile. If sand is slowly added to a heap on a table, the pile will evolve a critical slope. If it is too steep, a catastrophic avalanch will flatten it, and if it is too flat, the sand will gradually pile up to steepen the pile. Ultimately, the size of an avalanch produced by the random addition of an additional grain of sand will be unpredictable, giving rise to a power law distribution of avalanch sizes.

This concept of a dissipative system automatically becoming critical has been applied to many natural phenomena; indeed, it has been looked for in such diverse areas as earthquake structure [3] and economics. [4] The idea provides an alternative view of complex behavior in systems with many degrees of freedom. It complements the concept of "chaos," wherein simple systems with a small number of degrees of freedom can display quite complex behavior.

Ref. 2 introduced a simple cellular automaton model to illustrate self organized criticality. The model is formulated on a finite two dimensional square lattice with open boundaries. On each cell i of this lattice is a non-negative integer z_i representing the local slope of the sand. The dynamics of this system involves an instability threshold value for this slope, which I take to be $z_c = 4$. All cells in the system are updated simultaneously in discrete temporal steps. The updating rule is that if any cell has $z_i \geq z_c$, then that cell has its value decreased by four and each of its neighbors are increased by one. Such an event is referred to as a "tumbling," many of which can occur simultaneously in one updating step.

Note that in the interior of the lattice this dynamics locally conserves the total "sand" or sum of the z_i. Sand is lost only on the open boundaries. Because the dynamics involves a spreading of sand, it is easily shown that any configuration will eventually relax to a stable state with all slopes less than

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critical. Recently interesting geometric structures were seen to arise from the relaxation of uniform initial states [5].

After adding sand to the system for a while, there are some stable states which will never be obtained. For example, there is no way to completely clean the table of all sand giving the state with all slopes vanishing. For another example, two adjacent cells which are initially not empty can never be made so; when one tumbles to zero it adds a grain of sand to the other. The states which can be obtained from a full table have been called "recursive." [6] A recursive state is defined to be one that can be obtained by some addition of sand to any other state followed by relaxation to stability. One such state is the minimally stable state C^* defined as the state with all $z_i = z_c - 1$.

Ref. 6 showed that the number of recursive states is given by the determinant of the lattice Laplacian. Whereas an N site system has 4^N stable states, for large N the number of recursive states approaches $3.2102\ldots^N$. Ref. 6 also showed that in the self organized critical ensemble, each recursive state is equally likely; this ensemble serves as the analog of the Boltzmann distribution for a conventional statistical system.

Define a_i to represent the operation of adding a grain of sand to cell i followed by a relaxation of the system back to stability. Ref. 6 pointed out the remarkable fact that these operators all commute with eachother. The proof uses the linearity of toppling on the slopes z_i and uses the fact that a toppling only decreases the slope at the active site. For a detailed discussion see Ref. 1.

Several exact results follow from this observation. In particular, if we restrict ourselves to the recursive set of states, then these operators a_i have unique inverses. Thus, given a recursive configuration C, there is a unique recursive C' such that $a_i C' = C$. Because of this property the operations of adding sand generate an Abelian group.

I now define an operation of addition between states. Given stable states C and C' with corresponding slopes z_i and z'_i, I define the state $C \oplus C'$ to be that configuration obtained by relaxing the configuration with slopes $z_i + z'_i$. By construction, this definition is commutative.

Now consider restricting onself to recursive states. Since the process of adding one state to another can be decomposed into a set of individual sand additions, and because those additions are invertable on recursive states, this addition of states is itself invertable. Indeed, under $\oplus$ the recursive states themselves form an Abelian group, which is isomorphic to the group generated by the a_i.

One of the fundamental properties of any group is the existence of an identity element. Thus, among the recursive states there must exist a unique configuration I which when added to any other recursive state C relaxes back C. This is a property also possessed by the state with all $z_i = 0$, but that is not a recursive state.

Intrigued with the existence of this special state, I set out to find it. To proceed, I use the identity that adding four grains of sand to any site forces a tumbling and is equivalent to adding one grain to each neighbor. Thus the operation on a recursive state of adding four grains to one site and then removing one from each of its neighbors leaves the state unchanged. In terms of the operators a_i we have the statememt

$$\prod_{j \in n(i)} \left(a_i a_j^{-1} \right) = 1 \qquad (1)$$

where $n(i)$ denotes the nearest neighbors of site i.

Now consider applying this combined operation to all sites of the lattice. Any site in the interior will receive four grains but then have them taken away when the operation is applied to the neighbors.

Only at the edges will things not balance. Thus we are led to consider adding one grain to all edge sites and two to the corner sites. On any recursive state this addition will relax back to the starting state.

This argument leads me to consider the non-recursive state I_0 defined to have one on the edges, two on the corners, and zero elsewhere. This state, and any multiple of it, when added to a recursive state leaves that state unchanged. I now consider combining this state with itself iteratively until it becomes recursive. Thus I define $I_n = I_{n-1} \oplus I_{n-1}$. For large enough n I will have $I_n = I_{n-1} = I$, the identity I am searching for. Figure 1 shows the identity state constructed in this manner on a 188 by 288 cell lattice.

I now briefly mention a few other exact results for this model. First, if C is itself recursive, on adding I the amount of sand lost at the edges must equal the sand contained in I, and is independent of C. In fact, a much stronger result is true: the number of topplings on any given site i during this relaxation is separately independent of C.

Second, if we consider the critical ensemble where all recursive states are equally likely, the average number of tumblings at any site j resulting from the addition of one grain to site i is given by the simple expression

$$\langle T_{ij}(C) \rangle_{C \in \mathcal{R}} = \left(\Delta^{-1}\right)_{ij}. \qquad (2)$$

where the toppling matrix Δ is defined by

$$\Delta_{ij} = \begin{cases} +4 & i = j \\ -1 & i \text{ and } j \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

Note that this is a definition for a lattice Laplacian.

Finally, let me note a tenuous connection with Gribov copies. [7] In a rather elegant paper, B. Sharpe [8] considered a gauge fixing where the product of links coming out of any given site in a lattice gauge theory is constrained to be an arbitrary group element. For simplicity, let me consider the case of $U(1)$ and require the product of links coming out of a site to be unity. To find the number of Gribov copies of the vacuum, we put a gauge transforming phase g_i on each site and solve

$$\prod_{j \in n(i)} \left(g_i g_j^{-1}\right) = 1 \qquad (4)$$

where $n(i)$ denotes the nearest neighbors of site i. Note that this equation is identical to the relation between the operators a_i used in the above

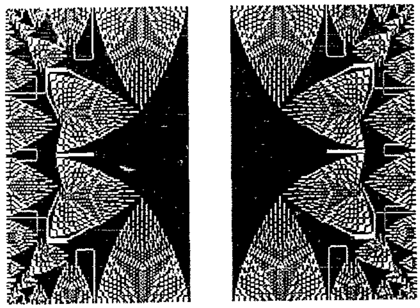

Fig. 1a. The first bit of the identity state on a 188 by 288 cell lattice. The image is black where this bit is set, i. e. for slopes of 1 or 3.

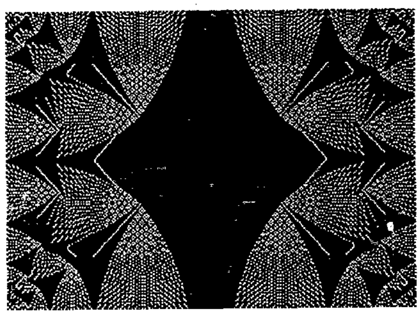

Fig. 1b. The second bit of the identity state on a 188 by 288 cell lattice. The image is black where this bit is set, i. e. for slopes of 2 or 3.

construction of the identity state. The number of solutions to this equation is the determinant of the matrix in Eq. (3) and equals the number of recursive states in the sandpile model.

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