

SELF ORGANIZED CRITICALITY

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ABSTRACT

Self organized criticality refers to the tendency of highly dissipative systems to drive themselves to a critical state. This has been proposed to explain why observed physics often displays a wide disparity of length and time scales. The phenomenon can be studied in simple cellular automaton models.

Many objects in nature are best described geometrically as fractals, with self-similar features on all length scales. The universe consists of clusters of galaxies, organized in clusters of clusters of galaxies and so on [1]. Mountain landscapes have peaks of all sizes, from kilometers down to millimeters. River networks consists of streams of all sizes. Turbulent fluids have vortices over a wide range of sizes. Earthquakes occur on structures of faults ranging from thousands of kilometers to centimeters.

The origin of fractals is a dynamical, not a geometrical, problem. The laws of physics are local, but fractals are nevertheless organized over the furthest distances. The mystery is enhanced by the fact that large equilibrium systems, operating near their ground state, tend to be only locally correlated. Only at a critical point where a continuous phase transition takes place are those systems fractal.

But real systems are dissipative, with friction, and rarely go to their ground state, unlike the ideals discussed in freshman physics. Consider, for example, a pendulum. The ideal motion is periodic and for small amplitudes is well approximated by a sine wave. To make it more realistic one can put in a drag term, giving rise to a damped oscillatory behavior with with a decreasing amplitude theoretically continuing forever. However, in the real world the motion will be impeded by some imperfections, perhaps in form of grit. Once the amplitude gets small enough, the pendulum will suddenly stop, and this will generally occur at the end of a swing where the velocity is smallest. This is not the state of smallest energy, and indeed the probability is a minimum for stopping at exactly the bottom of the potential. In a sense, the system is most likely to settle near a “minimally-stable” state, far from any “thermal equilibrium.”

Generalizing to a multi-dimensional system of many coupled pendula, a new issue arises. A minimally stable state will be particularly sensitive to small perturbations which can “avalanche” through the system. Thus, small disturbances could grow and propagate through the system with little resistance despite the damping

and impediments. Since energy is dissipated through this process, the energy must be replenished for avalanches to continue. The systems that we shall study are ones where energy is constantly supplied and eventually dissipated in the form of avalanches.

The canonical example is a simple pile of sand. Adding sand slowly to a flat pile will result only in some local rearrangement of particles. The individual grains, or degrees of freedom, do not interact over large distances. Continuing the process will result in the slope increasing to a critical value where an additional grain of sand gives rise to avalanches of any size, from a single grain falling up to the full size of the sand pile. The pile can no longer be described in terms of many local degrees of freedom, but only a holistic description in terms of one sandpile will do. The distribution of avalanches follows a power law.

“Self-Organized Criticality” refers to this tendency of large dissipative systems to drive themselves to a critical state with a wide range of length and time scales [2-5]. The idea provides a unifying concept for large scale behavior in systems with many degrees of freedom. It has been looked for in such diverse areas as earthquake structure, economics, and biological evolution.

Self-organized criticality complements the concept of “chaos” wherein simple systems with a small number of degrees of freedom can display quite complex behavior. Chaos is associated with fractal “strange” attractors in the phase space spanned by non-linear systems with only a few degrees of freedom. These self-similar structures need have little to do with fractals in real spatially extended physical systems. Specifically, chaotic systems exhibit white noise with short temporal correlations whereas fractal systems are expected to have long range temporal correlations. In contrast, self-organized criticality emphasizes unifying features in the coherent evolution of systems with many degrees of freedom.

The self-organized critical state can be demonstrated by computer simulations on toy sandpile models. The simplest example is a cellular automaton formulated on a two-dimensional regular lattice of N sites. Integer variables z_i on each site i are used to represent the local sandpile height. Here we consider a two-dimensional lattice with open boundaries. Addition of a sand particle to a site i is represented by increasing the value of z_i at that site by unity. When the height somewhere exceeds a critical value z_{cr} , here taken to be 3, there is a toppling event wherein 1 grain of sand is transferred from the unstable site to each of the 4 neighbor sites, i.e the value of z_i is reduced by 4 and the values of z at the 4 neighbor sites are increased by 1. The updating is done concurrently, with all sites updated simultaneously. The initial toppling may initiate a chain reaction, where the total amount of topplings is a measure of the size of an avalanche.

To explore self organized criticality in this model, one can randomly add sand and have the system relax. The result of such an addition becomes unpredictable, with one only being able to find the outcome by actually simulating the resulting avalanche. Figure 1 shows a log-log plot of the distribution of the avalanche sizes s

Fig. 1 A log-log plot of the distribution of the avalanche sizes s and durations t for the sandpile model.

and durations t . The linearity indicates a power law,

$$P(s) \sim s^{1-\tau}, \quad \tau \simeq 2.1, \quad (1)$$

where s is the number of tumblings in an avalanche and P is the probability distribution for avalanches of a given size.

For a random distribution of z 's one might expect the chain reaction generating the avalanche to be either sub-critical, in which case the avalanche would die after a few steps and large avalanches would be exponentially unlikely, or super-critical, in which case the avalanche would explode with a collapse of the entire system. The power law indicates that the reaction is precisely critical, i.e the probability that activity at some site branches into more than one active site is balanced by the probability that the activity dies. Thus, by evolving through avalanche after avalanche, the matrix has “learned” to respond critically to the next perturbation.

Several other quantities which obey fractal scaling laws can be defined for the sandpile. For instance, the duration t , that is the number of updatings for an avalanche to complete, has a distribution

$$P(t) \sim t^{1-\tau_t}, \quad \tau_t \simeq 2.14, \quad (2)$$

Also, the number of distinct tumbled sites, s_d , which is different from the total number of topplings since some sites topple more than once, goes as

$$P(s_d) \sim s_d^{1-\tau_d}, \quad \tau_d \simeq 2.07. \quad (3)$$

The values of the exponents quoted here were calculated by Kim Christensen [6]. The model can be defined in $d = 3, 4$, etc. dimensions. For instance, $\tau \simeq 2.31$ in

three dimensions; thus, the values of the exponents depend on d . For an excellent discussion of these exponents and their relation to each other, see the paper by Christensen *et al.* [7].

It would be highly desirable to have an analytical theory, such as the renormalization group theory for equilibrium critical phenomena, by which one could estimate the exponents and at the same time gain insights into the mechanisms of self-organized criticality. We are not yet at that point. However, in a series of papers, Deepak Dhar and co-workers have shown that the sand model has some rather remarkable mathematical properties [8-11]. In particular, the critical attractor of the system is characterized in terms of an Abelian group. The properties of the group can be utilized to calculate the number of states belonging to the critical attractor, and the rate of convergence to the attractor. Further consequences of the Abelian algebra have been explored [12-13].

Dhar introduced the useful toppling matrix $\Delta_{i,j}$ with integer elements representing the change in height, z at site i resulting from a toppling at site j [8]. Under a toppling at site j , the height at site i becomes $z_i - \Delta_{i,j}$. For the simple two dimensional sand model the toppling matrix is given as

$$\begin{aligned} \Delta_{i,j} &= 4 & i &= j \\ \Delta_{i,j} &= -1 & i, j & \text{ nearest neighbors} \\ \Delta_{i,j} &= 0 & & \text{ otherwise.} \end{aligned} \tag{4}$$

For this discussion there is little special to the specific lattice geometry; indeed the following results easily generalize to other lattices and dimensions; in fact, on a Cayley tree the model can be solved exactly. The analysis requires only that under a toppling of a single site i , that site has its slope decreased ($\Delta_{i,i} > 0$), the slope at any other site is either increased or unchanged ($\Delta_{i,j} \leq 0, j \neq i$), the total amount of sand in the system does not increase ($\sum_j \Delta_{i,j} > 0$), and, finally, that each site can be connected through topplings to some location where sand can be lost, such as at a boundary.

For the specific case in Eq. (4), the sum of slopes over all sites is conserved whenever a site away from the lattice edge undergoes a toppling. Only at the lattice boundaries can sand be lost. Thus the details of this model depend crucially on the boundaries, which we take to be open. A toppling at an edge loses one grain of sand and at a corner loses two.

The actual value of the threshold z_T is unimportant to the dynamics. This can be changed by simply adding constants to all the z_i . Thus without loss of generality we consider $z_T = 3$. With this convention, if all z_i are initially non-negative they will remain so, and we restrict ourselves to states C belonging to that set. The states where all z_i are positive and less than 4 are called stable; a state that has any z_i larger than or equal to 4 is called unstable. One conceptually important configuration is the minimally stable state C^* which has all the heights

at the critical value z_T . By construction, any addition of sand to C^* will give an unstable state.

We now formally define various operators acting on the states C . First, the “adding sand” operator α_i acting on any C yields the state $\alpha_i C$ where $z_i = z_i + 1$ and all other z are unchanged. Next, the toppling operator t_i transforms C into the state with heights z'_j where $z'_j = z_j - \Delta_{i,j}$. The operator U which updates the lattice one time step is now simply the product of t_i over all sites where the slope is unstable,

$$UC = \prod_i t_i^{p_i} C \quad (5)$$

where $p_i = 1$ if $z_i \geq 4$; 0 otherwise. Using U repeatedly we can define the relaxation operator R . Applied to any state C this corresponds to repeating U until no more z_i change. Neither U nor R have any effect on stable states. Finally, we define the avalanche operators a_i describing the action of adding a grain of sand followed by relaxation

$$a_i C = R \alpha_i C. \quad (6)$$

At this point it is not entirely clear that the operator R exists; that is it might be that the updating procedure enters a non-trivial cycle. We now prove that this is impossible. First note that a toppling in the interior of the lattice does not change the total amount of sand. A toppling on the boundary, however, decreases this sum due to sand falling off the edge. Thus, the total sand in the system is a non-increasing quantity. No cycle can have toppling at the boundary since this will decrease the sum. Next, the sand on the boundary will monotonically increase if there is any toppling one site away. This can not happen in a cycle, thus there can be no topplings one site away from the edges. By induction there can be no toppling arbitrary distances from the boundary; thus, there can be no cycle, and the relaxation operator exists. Note that for a general geometry this result requires that every site be eventually connected to an edge where sand can be lost. With periodic boundaries no sand would be lost and thus cycles are expected and observed. We call these unphysical models “Escher models” after the artist constructing drawings of water flowing perpetually downhill and yet circulating in the system.

It is useful to introduce the concept of recursive states. This set, denoted \mathcal{R} , includes those stable states which can be reached from any stable state by some addition of sand followed by relaxation. As the minimally stable state C^* can be obtained from any other state by adding just enough sand to each site to make z_i equal to three, it belongs to \mathcal{R} . Thus, one might conveniently define \mathcal{R} as the set of states which can be obtained from C^* by some addition of sand.

It is easily shown that there exist non-recursive, transient states; for instance, no recursive state can have two adjacent heights both being zero. One can also show that the self-organized critical ensemble, reached under random addition of sand to the system, has equal probability for each state in the recursive set.

The crucial result of Refs. [8-11] is that the operators a_i acting on stable states commute, and are invertable when restricted to recursive states. Indeed they generate an Abelian group when applied to recursive states.

This result enables us to count the number of recursive states. As all recursive states can be obtained by adding sand to C^* , we can write any state $C \in \mathcal{R}$ in the form

$$C = \left(\prod_i a_i^{n_i} \right) C^*. \quad (7)$$

Here the integers n_i represent the amount of sand to be added at the respective sites. However, in general there are several different ways to reach any given state. In particular, adding four grains of sand to any one site must force a toppling and is equivalent to adding a single grain to each of its neighbors. This can be expressed as the operator statement

$$a_i^4 = \prod_{j \in nn} a_j \quad (8)$$

where the product is over the nearest neighbors to site i . We can rewrite this equation by multiplying by the product of inverse avalanche operators on the nearest neighbors on both sides, thus obtaining

$$\prod_j a_j^{\Delta_{ij}} = E \quad (9)$$

where E is the identity operator. This allows us to change the powers appearing in Eq. (7). If we now label states by the vector $n = (n_1, n_2, n_3, \dots, n_N)$ we see that two such states are equivalent if the difference of these vectors is of the form $\sum_j \beta_j \Delta_{ij}$ where the coefficients β_j are integers. These are the only constraints; if two states can not be related by toppling they are independent. Thus any vector n can be translated repeatedly until it lies in an N - dimensional hyper-paralleliped whose base edges are the vectors Δ_{ji} , $j = 1, \dots, N$. The vertices of this object have integer coordinates and its volume is the number of integer coordinate points inside it. This volume is just the absolute value of the determinant of Δ . Thus the number of recursive states is given by the determinant of the toppling matrix Δ .

For large lattices this determinant can be found easily by Fourier transform. In particular, whereas there are $4N$ stable states, there are only

$$\exp \left(N \int_{(-\pi, -\pi)}^{(\pi, \pi)} \frac{d^2 q}{(2\pi)^2} \ln(4 - 2q_x - 2q_y) \right) \simeq (3.2102\dots)^N \quad (10)$$

recursive states. Thus starting from an arbitrary state and adding sand, the system “self-organizes” into an exponentially small subset of states forming the attractor of the dynamics.

Majumdar and Dhar [11] have constructed a simple “burning” algorithm to check and enumerate the configurations belonging to the recursive set. For a given

configuration, add one particle to each of the edge sites, two particles to the corners, and update according to the usual rules. If the original state is recursive, this will generate an avalanche under which each site of the system will tumble exactly once. If the state is not recursive, some untumbled sites will remain. Fig. 2 shows such a process underway on a typical recursive state. Here sites which have already burned are shown in cyan, while the remaining sites in the center have not yet burned. The small number of sites shown in light tan are the active active burning sites. Heights zero through three are shown as black, red, blue, and green, respectively. Note the fractal shape of the interphase.

One might ask if these power law distributions of avalanches can be observed with real sandpiles. After a couple of false starts with inconclusive results, there are now several experiments reporting power law distributions of avalanches [14-16]. Grumbacher *et al.* [15] built small heaps on a scale, and monitored the distribution of avalanches of particles falling off the edges. The experiments were performed using iron and glass spheres of the same size. In all cases a power law distribution function was found. In a remarkable experiment Bretz *et al.* [16] were even able to image the flow of small avalanches not reaching the edge and measure their flow and size.

Mandelbrot [1] has suggested that the dissipation of energy in turbulent systems is confined to a fractal structure with features at all length scales. This behavior can be simulated by a simple forest-fire model [17]. Distribute randomly a number of trees (green dots) and a number of fires (yellow dots). on a two dimensional rectangular lattice. Sites can also be empty. Update the system at each time t as follows: (1) grow new trees with probability p from sites that are empty at time $t - 1$; (2) trees that were on fire at time $t - 1$ die (become empty sites) and are removed at time t ; (3) a tree that has a fire as a nearest neighbor at time $t - 1$ catches fire at time t . Periodic boundaries are used. After a while the system evolves to a critical state with fire fronts of all sizes (Figure 3). Drossel and Schwabl [18] have extended the model by adding a small probability f of igniting new fires at each time step. In the limit $f/p \rightarrow 0$ the ignitions create forest fires where the number of trees burned, s , follows a power law, $P(s) \sim s^{1-\tau}$, with $\tau \simeq 2$.

The physics of earthquakes can be described in a similar language. The crust in a fault region is driven by imposing a force or a strain over a large length L . In the stationary state, the energy is dissipated in narrow fault structures forming a fractal set. The spatio-temporal correlation functions for the two phenomena are quite similar although the time scales are vastly different. In both cases, the energy enters the system uniformly (zero wave vector) and leaves the system locally. There are a lot of similarities between the earthquake model studied by Olami *et al.* [19] and the forest fire model. The analogy has been explored in some detail by Kagan [20]. Maybe it is useful to think of self-organized criticality and turbulence as one and the same phenomenon.

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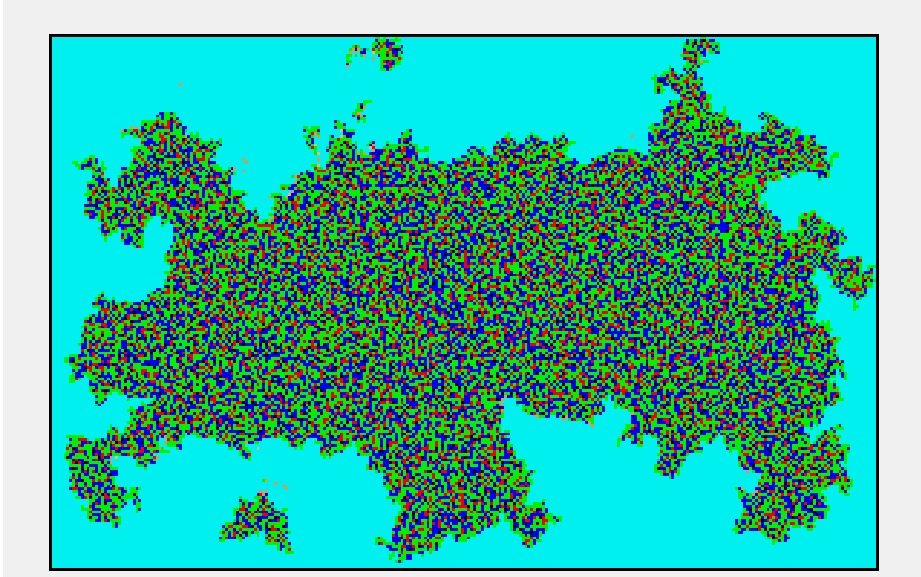


Fig. 2 The burning algorithm being applied to a typical critical state. Burnt sites are cyan, burning sites are tan, and heights zero through three are shown as black, red, blue, and green, respectively.

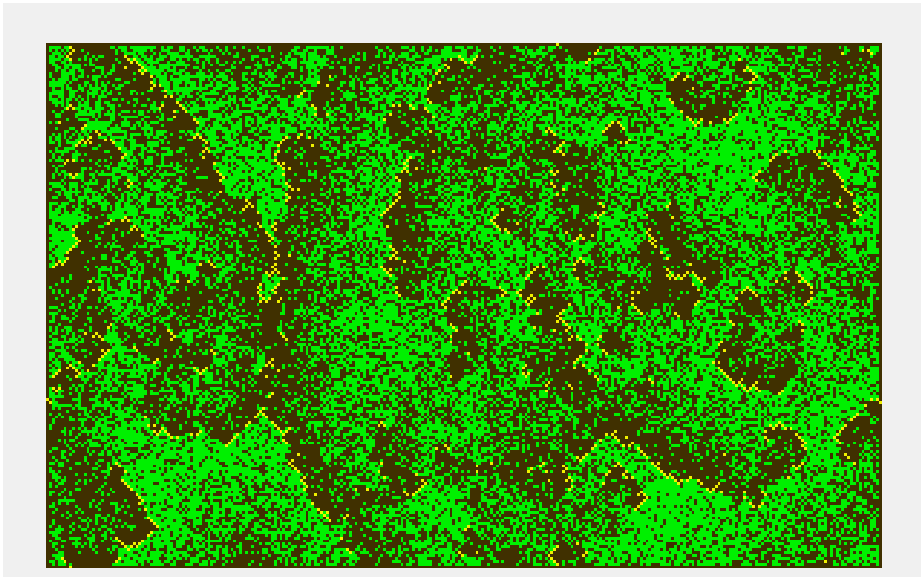


Fig. 3 A typical state in the dynamics of the forest fire model described in the text. Trees are green, fires yellow, and empty sites are black. The periodic lattice is 286 sites by 178 sites and new trees are born on empty sites with a probability of $1/32$ per time step.