# Species Doubling and Chiral Lagrangians

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Coupling gauge fields to the chiral currents from an effective Lagrangian for pseudoscalar mesons naturally gives rise to a species doubling phenomenon similar to that seen with fermionic fields in lattice gauge theory. [S0031-9007(96)00484-X]

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Species doubling is one of the oldest puzzles in lattice gauge theory. Naive fermion formulations on a lattice are plagued by the appearance of spurious low energy states. Various schemes have been implemented to remove the extra particles, but usually at the expense of mutilating valid symmetries. Only recently has a lattice formulation been presented that elegantly preserves the underlying chiral invariance of the strong interactions [1].

The problem is intricately entwined with the famous axial anomalies. A lattice regulator removes all infinities, so anomalies require an explicit symmetry breaking at the outset. This is also familiar from the Pauli-Villars [2] approach, where the mass of the heavy regulator field is not chirally symmetric. As the regulator is removed, it leaves a remnant determining an overall chiral phase. When the physical quarks maintain a finite mass, this phase is observable as the well-known strong *CP* parameter $\theta$ [3].

Going beyond purely hadronic physics to the gauge interactions of the electroweak theory, nonperturbative chiral issues remain unresolved. The $W$ bosons couple in an inherently parity violating manner, and rely for consistency on a subtle anomaly cancellation between quark and lepton contributions. While a flurry of recent work has advocated treating the fermions with a separate limit [4–6], it remains unknown how to implement this cancellation in a fully finite and gauge-invariant lattice theory. It is tempting to speculate that there is a shortcoming in either the lattice approach or in the standard model.

Here we argue that the so-called "doubling" problem is not unique to the lattice approach, but is a more general consequence of chiral anomalies. Starting with an effective Lagrangian for the strong interactions of the pseudoscalar mesons, we consider coupling gauge fields to the $\mathrm{SU}(n_f) \times \mathrm{SU}(n_f)$ symmetries of this model. When these gauge fields are themselves chiral, the gauging process naturally introduces additional Goldstone fields mirroring the original theory. As with the lattice doublers, the mirror fields cancel anomalies. The issue reduces to nonperturbatively removing the extra species when the original theory is anomaly free.

In the chiral Lagrangian approach, the effects of anomalies are summarized in a term discussed some time ago by Wess and Zumino [7], and later elucidated by Witten [8]. This coupling requires extending the fields into an internal space, only the boundary of which is relevant to the equations of motion. On adding a coupling to a local gauge field, however, the boundary can acquire additional contributions. The essence of this paper is that these are most naturally written in terms of doubler fields.

Reference [9] treats a canonical quantization of anomalous fermion theories. For consistency they couple the gauge fields to a Wess-Zumino term. While their starting point was rather different, they reach a similar conclusion that anomalous theories naturally lead to the introduction of new degrees of freedom.

To start, we briefly review the basic philosophy behind the effective Lagrangian approach. Our underlying theory contains a set of fermionic quark fields $\psi^a(x)$ interacting with non-Abelian gauge fields. Here we suppress all indices except flavor, represented by the index $a$, and space-time, represented by $x$. From $\psi$ we project out right- and left-handed parts, $\psi_R^a = \frac{1}{2}(1+\gamma_5)\psi^a$ and $\psi_L^a = \frac{1}{2}(1-\gamma_5)\psi^a$. For the purpose of this discussion we ignore fermion masses; such could be introduced, as in the standard model, via a Higgs mechanism.

The underlying quark-gluon theory with massless quarks is invariant under a global $\mathrm{SU}(n_f) \times \mathrm{SU}(n_f)$ symmetry, where $n_f$ represents the number of flavors. Under this, the quark fields transform as $\psi_L^a \to \psi^b g_L^{ba}$ and $\psi_R^a \to \psi^b g_R^{ba}$. Here $g_L$ and $g_R$ are elements of $\mathrm{SU}(n_f)$. Formally the classical Lagrangian is also invariant under a global $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry of phases for the left and right quark fields, but the axial part of the latter symmetry is broken by quantum effects, leaving just the vector U(1) of fermion number. While anomalies also play the key role in this breaking, that is not the subject of this paper.

In the conventional view, the axial part of the global chiral symmetry is spontaneously broken by the vacuum, resulting in $n_f^2 - 1$ Goldstone bosons and a remaining explicit $\mathrm{SU}(n_f)$ flavor symmetry. This is usually described via the composite field $\overline{\psi}_R^a \psi_L^b$ acquiring a vacuum expectation value. Without flavor breaking, one can use the chiral symmetry to pick a standard vacuum with, say, $\langle \overline{\psi}_R^a \psi_L^b \rangle = v\,\delta^{ab}$. Here the parameter $v$ determining the magnitude of the expectation value requires a renormalization scheme for precise definition. The vacuum is degenerate (after the usual extension of the quantum Hilbert space to a Banach space), and one could choose to replace $\delta^{ab}$ by an arbitrary element $g^{ab}$ of SU$(n_f)$. The basic idea of the effective Lagrangian is to promote this element into a local field $g(x)$. Slow variations of this field represent the Goldstone bosons arising from the degeneracy of the vacuum.

Equivalently, imagine integrating out the fermionic fields under a constraint $\langle \overline{\psi}_R^a \psi_L^b \rangle = v g^{ab}(x)$; ignore the massive modes associated with fluctuations in $v$, and use the resulting path integral to define an effective theory for $g(x) \in$ SU$(n_f)$. The chiral symmetry becomes an invariance under $g(x) \rightarrow g_L^\dagger g(x) g_R$ for arbitrary $g_L$ and $g_R$.

More quantitatively, the approach represents an expansion in powers of the momenta of the light particles [10]. The lowest order action contains the first term in this expansion

$$S_0 = \frac{F_\pi^2}{4} \int d_x^4 \, \mathrm{Tr}(\partial_\mu g \partial_\mu g^\dagger) \,. \tag{1}$$

The numerical constant $F_\pi$ sets the scale and has an experimental value around 93 MeV. To relate this to conventionally normalized pion fields, we define $g = \exp(i\pi \cdot \lambda / F_\pi)$ where the $n_f^2 - 1$ matrices $\lambda$ generate SU$(n_f)$ and are normalized $\mathrm{Tr}\lambda^\alpha \lambda^\beta = 2\delta^{\alpha\beta}$.

From this lowest order action, the equations of motion are $\partial_\mu J_{L,\mu}^\alpha = 0$, where the "left" current is

$$J_{L,\mu}^\alpha = \frac{iF_\pi^2}{4} \, \mathrm{Tr}\lambda^\alpha (\partial_\mu g) g^\dagger \,. \tag{2}$$

These equations have an equivalent form involving "right" currents $J_{R,\mu}^\alpha = \frac{iF_\pi^2}{4} \, \mathrm{Tr}\lambda^\alpha g^\dagger \partial_\mu g$. There is a vast literature about adding higher derivative terms to the above action [11]. This, however, is not what this paper is about.

We are interested in a special higher derivative coupling which is necessarily present and describes the effects of anomalies from the underlying quark fields. As is well known, this term is curious in that it cannot be written simply as an integral of a local expression in $g(x)$, even though the resulting contribution to the equations of motion is fully local [7,8,11]. Continuing to write the equations of motion in terms of a current, a possible addition which satisfies the required symmetries is

$$J_{L,\mu}^\alpha = \frac{iF_\pi^2}{4} \mathrm{Tr}\lambda^\alpha (\partial_\mu g) g^\dagger + \frac{in_c}{48\pi^2} \epsilon_{\mu\nu\rho\sigma} \, \mathrm{Tr}\lambda^\alpha (\partial_\nu g) g^\dagger (\partial_\rho g) g^\dagger (\partial_\sigma g) g^\dagger \,. \tag{3}$$

The equations of motion remain that the current be divergenceless, $\partial_\mu J_{L,\mu}^\alpha = 0$.

The addition in Eq. (3) is the simplest possible term involving the antisymmetric tensor, a shadow of the factors of $\gamma_5$ involved in the chiral anomalies. As is also well known [8], quantum mechanics requires the dimensionless coupling strength $n_c$ to be an integer corresponding to the number of degrees of freedom ("colors") in the underlying confining theory. Thus this term must indeed be present.

Continuing with this lightning review, we desire an action which generates the above equations of motion. This requires extending the field $g(x)$ beyond a simple mapping of space-time into the group. For this purpose, we introduce an auxiliary variable $s$ to interpolate between the field $g(x)$ and some fixed group element $g_0$. Thus we consider an extended field $h(x,s)$ satisfying $h(x,1) = g(x)$ and $h(x,0) = g_0$. This extension is not unique, but the final equations of motion are independent of the chosen path. We now write the action

$$S = \frac{F_\pi^2}{4} \int d^4x \, \mathrm{Tr}(\partial_\mu g \partial_\mu g^\dagger) + \frac{n_c}{240\pi^2} \int d^4x \int_0^1 ds \epsilon_{\alpha\beta\gamma\delta\rho} \, \mathrm{Tr} h_\alpha h_\beta h_\gamma h_\delta h_\rho \,. \tag{4}$$

Here we introduce the shorthand notation $h_\alpha = i(\partial_\alpha h) h^\dagger$ and regard $s$ as a fifth coordinate. The antisymmetric tensor satisfies $\epsilon_{1,2,3,4,5} = 1$.

To find the equations of motion, consider a small variation of $h(x,s)$. This can be shown to change the final integrand by a total divergence, which then integrates to a surface term. Working with either spherical or toroidal boundary conditions in the space-time directions, this surface involves only the boundaries of the $s$ integration. When $s = 0$, space-time derivatives acting on the constant matrix $g_0$ will vanish. The surface at $s = 1$ generates precisely the desired additional term in Eq. (3).

Geometrically, the last term in Eq. (4) is the volume of a piece of the $S_5$ sphere appearing in the structure of SU$(n_f)$ for $n_f \geq 3$. The mapping of four-dimensional space-time into the group surrounds this volume. Global chiral rotations merely shift this region around, leaving its numerical volume invariant. As emphasized by Witten [8], this volume is defined only up to a multiple of the total volume of the $S_5$ mapping into the gauge group. Different extensions into the $s$ coordinate can modify the above five-dimensional integral by an integer multiple of $480\pi^3$. To have a well-defined quantum theory, the action must be determined up to a multiple of $2\pi$. Thus the quantization of $n_c$ to an integer value occurs, much like the charge of a magnetic monopole.

Crucial to this discussion is the irrelevance of the starting group element $g_0$ and the lower end of the $s$ integration. The main point of this paper is to emphasize the difficulty of maintaining this condition when the symmetries become local. In particular, we want to extend the symmetry and allow $g_{R,L}$ to depend on the space-time coordinate $x$. As usual, this requires the introduction of local gauge fields. When we make the transformation $g(x) \to g_L^\dagger(x)g(x)g_R(x)$, derivatives of $g$ transform as

$$\partial_\mu g \longrightarrow g_L^\dagger\Big(\partial_\mu g - \partial_\mu g_L g_L^\dagger g + g\partial_\mu g_R g_R^\dagger\Big)g_R . \quad (5)$$

To compensate, we introduce left and right gauge fields transforming as

$$\begin{aligned} A_{L,\mu} &\longrightarrow g_L^\dagger A_{L,\mu} g_L + i g_L^\dagger \partial_\mu g_L , \\ A_{R,\mu} &\longrightarrow g_R^\dagger A_{R,\mu} g_R + i g_R^\dagger \partial_\mu g_R . \end{aligned} \quad (6)$$

Then the combination

$$D_\mu g = \partial_\mu g - iA_{L,\mu} g + igA_{R,\mu} \quad (7)$$

transforms nicely: $D_\mu g \to g_L^\dagger D_\mu g g_R$. If we make the generalized minimal replacement $\partial_\mu g \to D_\mu g$ in $S_0$, we find a gauge invariant action.

A problem arises when we go on to the Wess-Zumino term. We require a prescription for the gauge transformation on the interpolated group element $h(x,s)$. For this purpose, note a striking analogy with the domain wall approach to chiral fermions first promoted by Kaplan [12]. There an extra dimension was also introduced, with the fermions being surface modes bound to a four-dimensional interface. The usual approach to adding gauge fields involves, first, not giving the gauge fields a dependence on the extra coordinate, and second, forcing the component of the gauge field pointing in the extra dimension to vanish [13,14]. To be more precise, in terms of a five-dimensional gauge field, we take $A_\mu(x,s) = A(x)$ and $A_s = 0$ for both the left- and right-handed parts. Relaxing either of these would introduce extra degrees of freedom for which there is no desire. Thus the natural extension of the gauge transformation to all values of $s$ is to take $h(x,s) \to g_L^\dagger(x)h(x,s)g_R(x)$ with $g_{L,R}$ independent of $s$.

With this prescription for interpolating the gauge fields into the $s$ dimension, we replace the derivatives in the Wess-Zumino term with covariant derivatives, similar to Eq (7). This alone does not give equations of motion independent of the interpolation into the extra dimension. However, adding terms linear and quadratic in the gauge field strengths allows construction of a five-dimensional Wess-Zumino term for which variations are again a total derivative. The term is still ambiguous up to nonminimal coupling. For the photon, parity invariance uniquely fixes such terms, but this goes beyond the subject of this paper.

This procedure works well for a vectorlike gauge field, where we take $g_L(x) = g_R(x)$ and $A_L = A_R$. We could, for example, take $g_0$ to be the identity, and then the gauge transformation cancels out at $s = 0$. The approach gives the coupling of the photon field to the pseudoscalar mesons, including [8] a piece that describes $\pi \to 2\gamma$. This supports the necessity of the Wess-Zumino term, and determines the coefficient to be proportional to the dimension of the quark representation in the underlying confining symmetry group.

A difficulty arises when coupling a gauge field to an axial current. For example, the weak bosons of the standard electroweak theory involve such a coupling. In this case the above prescription at $s = 0$ takes $g_0 \to g_L^\dagger(x)g_0 g_R(x)$, which in general will no longer be a constant group element. After a gauge transformation, variations of the action give new nonvanishing contributions to the equations of motion from the lower end of the $s$ integration.

The simplest solution promotes the $s = 0$ fields to be dynamical. Thus we replace the field $g(x)$ with two fields $g_0(x)$ and $g_1(x)$. The interpolating field now has the properties $h(x,0) = g_0(x)$ and $h(x,1) = g_1(x)$. The action becomes

$$S = \frac{F_\pi^2}{4}\int d^4x \,\mathrm{Tr}(D_\mu g_0 D_\mu g_0^\dagger + D_\mu g_1 D_\mu g_1^\dagger) + \Gamma , \quad (8)$$

where $\Gamma$ denotes the appropriately gauged Wess-Zumino term.

While we now have a gauge invariant theory, it differs from the starting theory through doubling of meson species. The extra particles are associated with the second set of group valued fields $g_0(x)$. The Wess-Zumino term of the new fields has the opposite sign since it comes from the lower end of the $s$ integration. Thus these "mirror" particles have reflected chiral properties and implement a cancellation of all anomalies. In essence, we have circumvented the subtleties in gauging the model.

The value of $F_\pi$ need not be the same for $g_0$ and $g_1$, so their strong interactions might differ in scale. Nevertheless, coupling with equal magnitude to the gauge bosons, the new fields cannot be ignored. The doubling is less severe than in the lattice approach, where each pairing in the number of fermion fields gives a factor of four in boson species.

Had we only coupled gauge fields to vector currents, we could easily remove the doublers using a diagonal mass term at $s = 0$. For example, with a term $M\,\mathrm{Tr}g_0(x)$ added to the Lagrangian density, $M$ could be arbitrarily large, forcing $g_0$ towards the identity. Such a term is invariant under vector rotations, but not under axial symmetries.

The doublers arise in complete analogy to the problems appearing in the surface mode approach to chiral lattice fermions [12–15]. In both cases, an extension to an extra dimension is introduced. Difficulties arise from the appearance of an extra interface in the $s$ coordinate. This new surface cannot be ignored since it couples with equal strength to the gauge fields.

If we relax the constraints and let $g_{L,R}$ depend on $s$, we expect problems similar to those seen with domain wall fermions. In particular, when the gauge fields vary in the extra dimension, four-dimensional gauge invariance is lost. Symmetry can be restored via a Higgs field, but this introduces the possibility of unwanted degrees of freedom in the physical spectrum. Reference [14] explores the possibility of sharply truncating the gauge field at an intermediate value of the extra coordinate. This gives rise to new low energy bound states acting much like the undesired doubler states.

Introducing a Higgs field does permit different masses for the extra species. In particular, the matter couplings of the Higgs field can depend on $s$. Qualitative arguments suggest that triviality effects on such couplings limit their strength, precluding masses for the extra species beyond a typical weak interaction scale. Presumably such constraints will be the strongest when the anomalies in the undoubled sector are not properly canceled. With domain-wall fermions, taking the Higgs-fermion coupling to infinity on one wall introduces a plethora of new low energy bound states [15].

These problems reemphasize the subtle way the standard model cancels anomalies between the quarks and the leptons. If the contributions of the leptons are ignored, no nonperturbative approach can be expected to accommodate gauged weak currents. Indeed, the doubling discussed here arises as a necessary consequence of residual anomalies. When the required cancellations occur between different fermion representations, perturbation theory appears to be consistent, while all known nonperturbative approaches remain awkward.

There are several possible solutions to these doubling problems. Least interesting would be some trivial missed issue in our search for a nonperturbative definition of a chiral gauge theory. On the other hand, mirror particles might actually exist, perhaps with masses comparable to the weak scale [16]. Such extra fields might even be useful in the spontaneous breaking of the electroweak theory [17]. A related alternative has the standard model arise from the spontaneous breaking of an underlying vectorlike unified theory containing additional heavy bosons coupling with opposite parity fermions [18]. All of these involve a profusion of new particles awaiting discovery. A speculative solution would twist the extra dimension so that the doubling particles could be among those already observed. This requires the interpolation in the extra dimension to mix the quarks and the leptons, all of which are involved in the anomaly cancellations of the standard model. While such a scheme remains elusive, it conceivably could require the existence of multiple families.

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