

Chiral Symmetry Versus the Lattice

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Abstract. After mentioning some of the difficulties arising in lattice gauge theory from chiral symmetry, I discuss one of the recent attempts to resolve these issues using fermionic surface states in an extra space-time dimension. This picture can be understood in terms of end states on a simple ladder molecule.

Chiral symmetry and lattice gauge theory provide two of the best known approaches to understanding non-perturbative phenomena in relativistic quantum field theory. However, rather interesting clashes appear when these methods are considered together. Our understanding of this problem has seen considerable progress in the last few years, although numerous unanswered questions remain. The purpose of this talk is to introduce some of these issues from a general point of view, avoiding technical details. For a more extensive reviews, see [1] and [2].

The issues involved are rather old, going back to the species doubling phenomena observed with the first papers on lattice gauge theory. To solve this doubling, the regulator was modified, but this modification directly breaks chiral symmetry. This feature is not a nemesis, but a virtue of the formalism. Without such modifications, there would be no room for the well known chiral anomalies to appear. Indeed, I believe that the conflict between chiral symmetry and the lattice is telling us something deep about the structure of relativistic quantum field theory.

I begin with a brief reminder of what lattice gauge theory is all about. Basically, it is nothing but a mathematical trick. By removing the infinities of the underlying field theory, the lattice gives us a well defined mathematical system independent of perturbative expansion. In this approach the world lines of a particle are replaced by discrete hops on a four dimensional lattice, as sketched in Fig. 1.

The lattice spacing a is an artificial construct and we must always keep in mind the need to take $a \rightarrow 0$ for physical results. While in place, however, the lattice provides an ultraviolet cutoff at momentum $\Lambda = \pi/a$. In addition to making the theory finite, the lattice enables Monte Carlo simulations, which currently dominate the field.

Since the lattice is a first principles approach to field theory, one could ask why care about the details of chiral symmetry. Just put the problem on the computer, predict particle properties, and they should come out correctly

if the underlying dynamics is relevant. While this may perhaps be a logical point of view, it ignores a vast lore built up over the years. In the context of the strong interactions, the pion and the rho mesons are made of the same quarks, the only difference being whether the spins are anti-parallel or parallel. Yet the pion, at 140 MeV, weighs substantially less than the 770 MeV rho. Chiral symmetry is at the core of the conventional explanation. Since the up and down quarks are fairly light, we have an approximately conserved axial vector current, and the pion is believed to be the remnant Goldstone boson of a spontaneous breaking of this chiral symmetry.

Another motivation for studying chiral issues arises when considering the weak interactions. Here we are immediately faced with the experimental observation of parity violation, neutrinos are left handed. In the standard electroweak model fundamental gauge fields are coupled directly to chiral currents. The corresponding symmetries are gauged, *i.e.* they become local, and are crucial to the basic structure of the theory. Since the lattice is the one truly non-perturbative regulator for defining a field theory, if one cannot find a lattice regularization for the standard model, the standard model itself may not be well defined.

A third reason to explore chiral symmetries comes from unified field theories. These usually have a large natural scale. In comparison, quark and lepton masses are much smaller. In such models chiral symmetry can protect fermion masses from large renormalization. This is also one of the prime reasons for the popularity of super-symmetry, which extends this protection to scalar particles, such as the Higgs meson.

The word “chiral,” based on the Greek word for hand, was introduced into modern scientific jargon by Lord Kelvin [3] in 1904 when in a rewriting of his Baltimore lectures he said “I call any geometrical figure, or group of points, chiral, and say it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself.”

The concept of chirality is most frequently used by chemists. Molecules whose structure is different from their mirror image are called chiral. For the

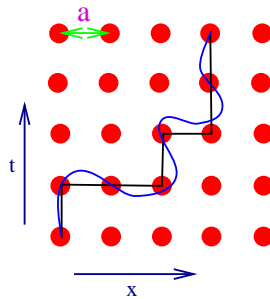


Fig. 1. Lattice gauge theory begins by approximating continuous space time with a discrete set of points

particle theorist, however, the use of this term is associated with subtleties of the Lorentz group and massless particles. When a particle is massless it travels at the speed of light. This is a limiting velocity for any observer, who cannot go faster than such a particle to reverse its direction. A direct consequence for particles with spin is that their helicity, i.e. angular momentum along their direction of motion, is frame invariant. For spin 1/2 fermions, the left and right handed components, ψ_L and ψ_R become independent fields. This independence is naively preserved under gauge interactions; a relativistic electron tends to preserve its helicity as it travels through electromagnetic fields.

This concept, however, is clouded by the so-called “chiral anomalies” [4]. In particular, the famous triangle diagram, sketched in Fig. 2, coupling two vector and one axial vector current is divergent, and no regularization can keep them both conserved. If either is coupled to a gauge field, such as electromagnetism, this diagram must be regulated with that particular current being conserved. Then the other cannot be. These anomalies are at the core of the lattice problems.

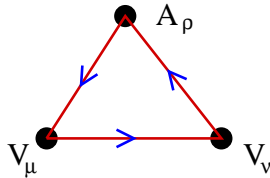


Fig. 2. The triangle diagram cannot be regulated so both vector and axial vector currents are conserved.

In one spatial dimension chirality reduces to separating particles into left and right movers. In this case the anomaly is easily understood via simple band theory [5]. A particle of non-zero mass m and momentum p has energy $E = \pm\sqrt{p^2 + m^2}$. Here I use a Dirac sea description where the negative energy states are filled in the normal vacuum. Considering the positive and negative energy states together, the spectrum has a gap equal to twice the particle mass. In the vacuum the Fermi level is at zero energy, exactly in the center of this gap. In conventional band theory language, the vacuum is an insulator.

In contrast, for massless particles where $E = \pm|p|$, the gap vanishes. The system becomes a conductor, as sketched in Fig. 3. Of course, conductors can carry currents, and here the current is proportional to the number of right moving particles minus the number of left movers. If we consider gauge fields, they can induce currents, a process under which the number of right or left movers cannot be separately invariant. This is the anomaly, without which transformers would not work.

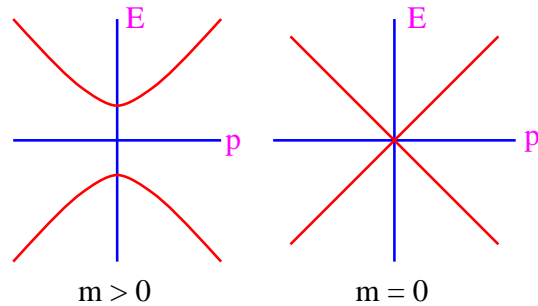


Fig. 3. In one dimension the spectrum of massive particles has a gap, and the vacuum can be regarded as an insulator. The massless case, in contrast, represents a conductor. The anomaly manifests itself in the ability to induce currents in a wire.

This induction of currents is not a conversion of particles directly from left into right movers, but rather a sliding of levels in and out of the infinite Dirac sea. The generalization of this discussion to three spatial dimensions uses Landau levels in a magnetic field; the lowest Landau level behaves exactly as the above one dimensional case [5].

One particularly intriguing consequence for the standard model is that baryon number is an anomalous charge. Indeed, 't Hooft[6] pointed out a specific baryon-number-changing mechanism through topologically non-trivial gauge configurations. The rate is highly suppressed due to a small tunneling factor and is far too small to observe experimentally. Nevertheless, the process is there in principle, and any valid non-perturbative formulation of the standard model must accommodate it. If we have a fully finite and exactly gauge invariant lattice theory, the dynamics must contain terms which violate baryon number. This point was emphasized some time ago by Eichten and Preskill [7] and further by Banks [8].

Without baryon violating terms, something must fail. In naive approaches to lattice fermions the problem materializes via extra particles, the so-called doublers, which cancel the anomalies. For the strong interactions alone, a vector-like theory, Wilson [9] showed how to remove the doublers by adding a chirally non-symmetric term. This term formally vanishes in the continuum limit, but serves to give the doublers masses of order the inverse lattice spacing. As chiral symmetry is explicitly broken, the chiral limit of vanishing pion mass is only obtained with a fine tuning of the quark mass. This is no longer “protected”; the bare and physical quark masses no longer vanish together. This approach works well for the strong interactions, but explicitly breaks a chirally coupled gauge theory. This entails an infinite number of gauge variant counter-terms to restore gauged chiral symmetries in the continuum limit [10]. It is these features that drive us to search for a more elegant formulation.

To proceed I frame the discussion in terms of extra space-time dimensions. The idea of adding unobserved dimensions is an old one in theoretical

physics, going back to Kaluza and Klein [11], and often is quite useful in unifying different interactions. Of course the extension of space-time to higher dimensions is crucial to modern string theories. There are probably further unexploited analogies here, but chiral symmetry in particular can become quite natural when formulated on higher dimensional membranes. Here I use only the simplest extension, involving one extra dimension.

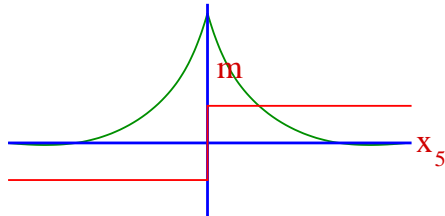


Fig. 4. A step in a five dimensional fermion mass can give rise to topological zero-energy fermion modes bound to a four dimensional interface.

I start with an observation of Callan and Harvey [12], building on Jackiw and Rebbi [13]. They argue that a five dimensional massive fermion theory formulated with an interface where the fermion mass changes sign, as sketched in Fig. 4, can give rise to a four dimensional theory of massless fermionic modes bound to the interface. The low energy states on the interface are naturally chiral, and anomalous currents are elegantly described in terms of a flow into the fifth dimension.

While the Callan and Harvey discussion is set in the continuum, Kaplan [14] suggested carrying the formalism directly over to the lattice. In the Wilson formulation, the particle mass is controlled via the hopping parameter, usually denoted K . The massless situation is obtained at a critical hopping, K_c , the numerical value of which depends on the gauge coupling. Thus, to set up an interface as used by Callan and Harvey, one should consider a five dimensional theory with a hopping parameter which depends on the extra fifth coordinate. This dependence should be constructed to generate a four dimensional interface separating a region with $K > K_c$ from one with $K < K_c$. Shamir [15] observed a substantial simplification on the $K < K_c$ side by putting $K = 0$. Then that region decouples, and the picture reduces to a four dimensional surface of a five dimensional crystal. The physical picture is sketched in Fig. 5. For a Hamiltonian discussion, see Ref. [16]. Indeed, surface modes are not a particularly new concept; in 1939 Shockley [17] discussed their appearance in band models when the inter-band coupling becomes strong. This approach has stimulated several closely related variations that have attracted considerable recent attention [1] [2] [18].

I will now discuss these “domain-wall fermions” from a rather unconventional direction. Following a recent paper of mine [19], I present the subject

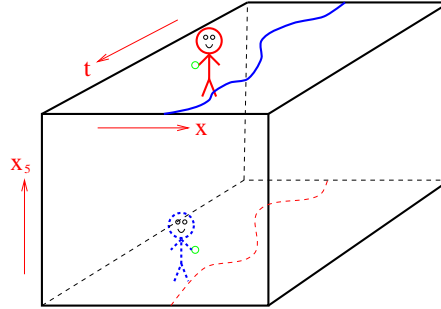


Fig. 5. Regarding our four dimensional world as a surface in five dimensions.

from a “chemists” point of view, in terms of a chain molecule with special electronic states carrying energies fixed by symmetries. For lattice gauge theory, placing one of these molecules at each space-time site gives excitations of naturally zero mass. This is in direct analogy to the role of chiral symmetry in conventional continuum descriptions. After presenting this picture, I will wander into some comments and speculations about exact lattice chiral symmetries and schemes for gauging them.

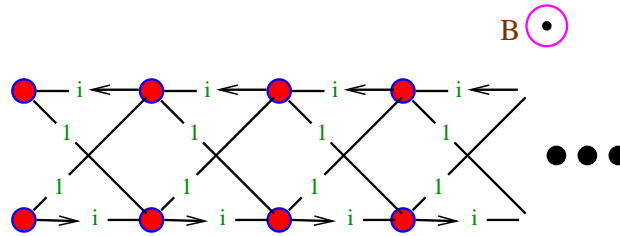


Fig. 6. The basic cross linked lattice in a magnetic field. The numbers on the bonds represent phases giving half a unit of flux per plaquette. If we slightly slope the vertical bonds alternately in and out of the plane, the model is a chain of tetrahedra, linked on opposite edges.

To start, consider two rows of atoms connected by horizontal and diagonal bonds, as illustrated in Fig. 6. The bonds represent hopping terms, wherein an electron moves from one site to another via a creation-annihilation operator pair in the Hamiltonian. Later I will include vertical bonds, but for now consider just the horizontal and diagonal connections.

Years ago during a course on quantum mechanics, I heard Feynman present an amusing description of an electron’s behavior when inserted into a lattice. If you place it initially on a single atom, the wave function will gradually spread through the lattice, much like water poured in a cell of a metal ice cube tray. With damping, it settles into the ground state which has equal amplitude on each atom. To this day I cannot fill an ice cube tray

without thinking of this analogy and pouring all the incoming water into a single cell.

I now complicate this picture with a magnetic field applied orthogonal to the plane of the system. This introduces phases as the electron hops, causing interesting interference effects. In particular, consider a field of one-half flux unit per plaquette. This means that when a particle hops around a unit area (in terms of the basic lattice spacing) the wave function picks up a minus sign. Just where the phases appear is a gauge dependent convention; only the total phase around a closed loop is physical. One choice for these phases is indicated by the numbers on the bonds in Fig. 6.

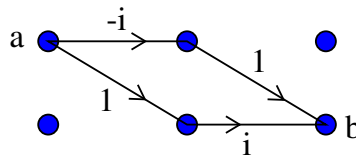


Fig. 7. With half a unit of magnetic flux per plaquette, the paths for an electron to move two sites interfere destructively. A particle on site *a* cannot reach *b*.

The phase factors cause cancellations and slow diffusion. For example, consider the two shortest paths between the sites *a* and *b* in Fig. 7. With the chosen flux, these paths exactly cancel. For the full molecule this cancellation extends to all paths between these sites. An electron placed on site *a* can never diffuse to site *b*. Unlike in the ice tray analogy, the wave function will not spread to any site beyond the five nearest neighbors.

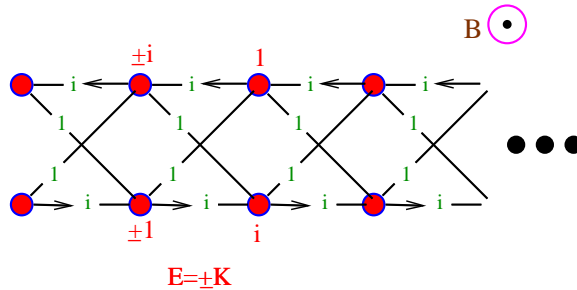


Fig. 8. Two localized energy eigenstates occur on every plaquette of the molecule.

As a consequence, the Hamiltonian has localized eigenstates. While it is perhaps a bit of a misuse of the term, these states are “soliton-like” in that they just sit there and do not change their shape. There are two such states per plaquette; one possible representation for these two states is shown in Fig. 8. The states are restricted to the four sites labeled by their relative

wave functions. Their energies are fixed by the size of the hopping parameter K .

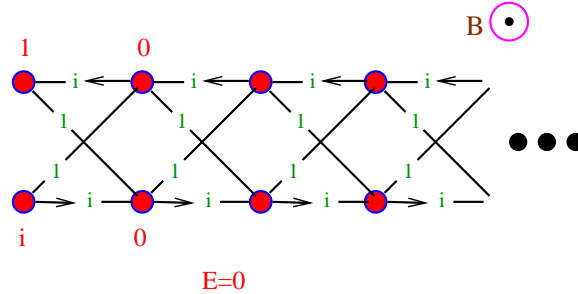


Fig. 9. A zero energy state bound to the lattice end.

For a finite chain of length L there are $2L$ atoms, and thus there should be a total of $2L$ possible states for our electron (ignoring spin). There are $L - 1$ plaquettes, and thus $2L - 2$ of the above soliton states. This is almost the entire spectrum of the Hamiltonian, but two states are left over. These are zero energy states bound to the ends of the system. The wave function for one of those is shown in Fig. 9. We now have the full spectrum of the Hamiltonian: $L - 1$ degenerate states of positive energy, a similar number of degenerate negative energy states, and two states of zero energy bound on the ends.

Now consider what happens when vertical bonds are included in our molecule. The phase cancellations are no longer complete and the solitonic states spread to form two bands, one with positive and one with negative energy. However, for our purposes, the remarkable result is that the zero modes bound on the ends of the chain are robust. The corresponding wave functions are no longer exactly located on the last atomic pair, but now have an exponentially suppressed penetration into the chain. Fig. 10 shows the wave function for one of these states when the vertical bond has the same strength as the others. There is a corresponding state on the other end of the molecule.

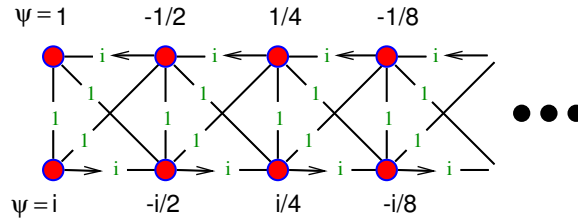


Fig. 10. The zero energy state is robust under adding vertical bonds.

When the chain is very long, both of the end states are forced to zero energy by symmetry considerations. First, since nothing distinguishes one end of the chain from the other, they must have equal energy, $E_L = E_R$. On the other hand, a change in phase conventions, effectively a gauge change, can change the sign of all the vertical and diagonal bonds. Following this with a left right flip of the molecule will change the signs of the horizontal bonds. This takes the Hamiltonian to its negative, and shows that the states must have opposite energies, $E_L = -E_R$. This is indicative of a particle-hole symmetry. The combination of these results forces the end states to zero energy, with no fine tuning of parameters.

For a finite chain, the exponentially decreasing penetration of the end states into the molecule induces a small interaction between them. They mix slightly to acquire exponentially small energies $E \sim \pm e^{-\alpha L}$. As the strength of the vertical bonds increases, so does the penetration of the end states. At a critical strength, the mixing becomes sufficient that the zero modes blend into the positive and negative energy bands. In the full model, the mixing depends on the physical momentum, and this disappearance of the zero modes is the mechanism that removes the “doublers” when spatial momentum components are near π in lattice units [16].

Energy levels forced to zero by symmetry lie at the core of the domain wall fermion idea. On every spatial site of a three dimensional lattice we place one of these chain molecules. The distance along the chain is usually referred to as a fictitious “fifth” dimension. The different spatial sites are coupled, allowing particles in the zero modes to move around. These are the physical fermions. The symmetries that protect the zero modes now protect the masses of these particles. Their masses receive no additive renormalization, exactly the consequence of chiral symmetry in the continuum. The physical picture is cartooned in Fig. 11, where I have rotated the fifth dimension to the vertical. Our world lines traverse the four dimensional surface of this five dimensional manifold.

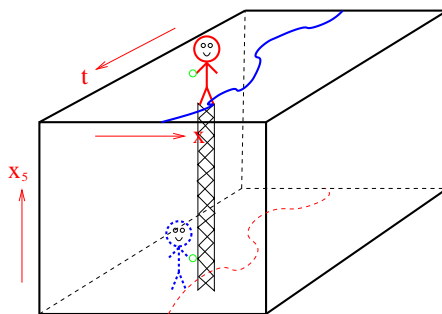


Fig. 11. The zero modes of the chain molecules become the quarks of which we are made.

This scheme is for the fermions of the theory, and nothing extra is needed for the gauge fields. Indeed, we do not want the gauge fields to see the extra dimension. Thus we keep $A(x_\mu, x_5) = A(x_\mu)$ independent of x_5 and have no fifth component, i.e. $A_5 = 0$. In some sense calling our extra coordinate a dimension is a bit of a convention; x_5 might as well be regarded as a “flavor” [18].

The domain wall approach gives rise to a natural chiral theory on one wall. This gives a particularly elegant formulation of the strong interactions, minimizing the doubling required by existing no-go theorems. In this picture the left and right handed quarks reside on opposite walls.

For a chiral theory, however, the existence of anti-walls raises unresolved questions. For a finite fifth dimension the walls always appear in pairs. Because the gauge fields do not know about the fifth dimension, the same gauge fields appear on each wall. The opposite chirality fermion zero modes found there represent “mirror” fermions; a theory with a left handed neutrino on one wall will naturally have a right handed partner on the other. How to resolve this issue for the standard model is still controversial.

One speculative approach was presented a few years ago [21], where an unusual identification of the particles on the two walls was enabled via the introduction of a four fermion coupling deep in the interior of the extra dimension, as sketched in Fig. 12. The introduced four-fermion operator is “technically irrelevant,” and fully gauge invariant. It is baryon number violating, but, as noted earlier, this is a necessary feature of any fully finite formulation of the standard model.

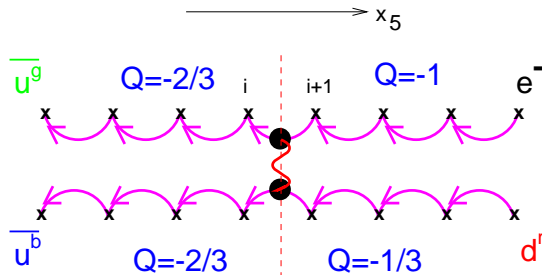


Fig. 12. Introducing a charge transfer involving four fermionic fields gives rise to a possible scheme for putting the standard model on the lattice.

This particular approach has not received much attention because of difficulties in treating the four fermion coupling. In particular, there is a serious danger that such a coupling could induce a spontaneous breaking of one of the gauge symmetries. This would be a disaster for the picture since such breaking would naturally be at the scale of the cutoff.

I hope this description of domain-wall fermions in terms of simple chain molecules has at least been thought provoking. I now ramble on with some

general remarks about the basic scheme. The existence of the end states relies on using open boundary conditions in the fifth direction. If we were to curl our extra dimension into a circle, they will be lost. To retrieve them, consider cutting such a circle, as in Fig. 13. Of course, if the size of the extra dimension is finite, the modes mix slightly. This is crucial for the scheme to accommodate anomalies [16].

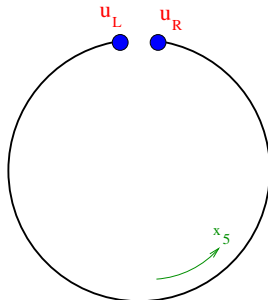


Fig. 13. A compact fifth dimension must be cut in order to generate the chiral zero modes of the domain-wall formalism.

Suppose I want a theory with two flavors of light fermion, such as the up and down quarks. For this one might cut the circle twice, as shown in Fig. 14. Remarkably, this construction keeps one chiral symmetry exact, even if the size of the fifth dimension is finite. Since the cutting divides the molecule into two completely disconnected pieces, in the notation of the figure we have the number of $u_L + d_R$ particles absolutely conserved. Similarly with $u_R + d_L$. Subtracting, we discover an exactly conserved axial charge corresponding to the continuum current

$$j_{\mu 5}^3 = \bar{\psi} \gamma_\mu \gamma_5 \tau^3 \psi$$

The conservation holds even with finite L_5 . There is a small flavor breaking since the u_L mixes with the d_R . These symmetries are reminiscent of Kogut-Susskind [20], or staggered, fermions, where a single exact chiral symmetry is accompanied by a small flavor breaking. Now, however, the extra dimension gives additional control over the latter.

Despite this analogy, the situation is physically somewhat different in the zero applied mass limit. Staggered fermions are expected to give rise to a single zero mass Goldstone pion, with the other pions acquiring mass through the flavor breaking terms. In my double cut domain-wall picture, however, the zero mass limit has three degenerate equal mass particles as the lowest states. To see how this works it is simplest to discuss the physics in a chiral Lagrangian language. The finite fifth dimension generates an effective mass term, but it is not in a flavor singlet direction. It is in a flavor direction orthogonal to the naive applied mass. In the usual Mexican hat picture,

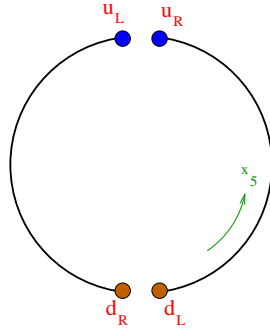


Fig. 14. Cutting the compact fifth dimension twice gives two flavors of fermion. With the identifications here, one flavored chiral symmetry is exact, even when the lattice spacing and the size of the fifth dimension are finite.

the two mass terms compete and the true vacuum rotates around from the conventional “sigma” direction to the “pi” direction.

Now I become more speculative. The idea of using multiple cuts in the fifth dimension to obtain several species suggests extensions to zero modes on more complicated manifolds. By having a variety of zero modes, we have a mechanism to generate multiple flavors. Maybe all the physical fermions in four dimensions arise from a single fermion field in the underlying higher dimensional theory. Schematically we might have something like shown in Fig. . where each point represents some four dimensional surface and the question remark represents structures in the higher dimension that need specification.

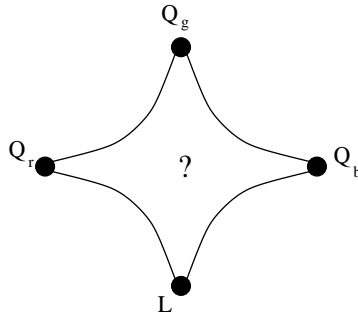


Fig. 15. Perhaps all fermions are special modes of a single higher-dimensional field. Here the three quark fields might represent different values of the internal $SU(3)$ symmetry, and L could represent a lepton from the same family.

One nice feature provided by such a scheme is a possible mechanism for the transfer of various quantum numbers involved in anomalous processes. For example, the baryon non-conserving 't Hooft process[6] might arise from a lepton flavor tunneling into the higher manifold and reappearing on another

surface as a baryon. This generic mechanism is in fact the basis of the specific proposed formulation of the standard model on the lattice[21] mentioned earlier.

To summarize, I have argued that because it is totally finite, the lattice forces honesty in understanding any peculiar phenomena that arises, and this can reveal deep features of quantum field theory. Chiral symmetry issues represent a dramatic example of this.

I presented a simple molecular picture for zero modes protected by symmetry. This illustrates the mechanism for mass protection in the domain-wall formulation of lattice fermions. Finally I speculated on schemes for generating multiple fermion species from the geometry of higher dimensional models. The latter may have connections with the activities in string theory.

Acknowledgment

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