arXiv:hep-lat/0110009 v1 4 Oct 2001

A new fermion Hamiltonian for lattice gauge theory

Michael Creutz [a] *, Ivan Horváth [b]†, and Herbert Neuberger [c] ‡

[a]Physics Department, Brookhaven National Laboratory,
Upton, NY 11973, USA

[b]Dept. of Physics and Astronomy, University of Kentucky,
Lexington, KY 40506, USA

[c]Dept. of Physics and Astronomy, Rutgers University,
Piscataway, NJ 08855, USA

We formulate Hamiltonian vector-like lattice gauge theory using the overlap formula for the spatial fermionic part, H_f. We define a chiral charge, Q_5 which commutes with H_f, but not with the electric field term. There is an interesting relation between the chiral charge and the fermion energy with consequences for chiral anomalies.

Lattice gauge theory and chiral symmetry represent two venerable non-perturbative approaches in particle theory. Historically, however, they have not easily meshed. In particular, the old species doubling problem forces one to insert chiral symmetry breaking somewhere into the regulator. The issues involved are deeply entwined with anomalies [1]. Stimulated by the domain wall fermion idea of Kaplan [2] and the overlap formulation of Neuberger and Narayanan [3], this topic has recently seen renewed attention and dramatic progress.

Here we explore an adaptation of the overlap idea to the Hamiltonian formalism. We find an interesting operator structure and a simple intuitive picture of how anomalies work in terms of a fermion eigenvalue flow.

To start, consider the conventional "continuum" Hamiltonian for a gauge theory

$$\begin{aligned} &H = H_f + H_g \\ &H_f = \overline{\psi} D_c \psi = \psi^\dagger \gamma_0 D_c \psi \\ &H_g = E^2 + B^2 \end{aligned} \tag{1}$$

where the fermion fields ψ anticommute and

$$D_c = \vec{\gamma} \cdot (\vec{\partial} + ig\vec{A}) + m. \tag{2}$$

The operator D_c consists of an antihermitean kinetic piece and the hermitean mass term (we use hermitean gamma matrices). The antihermitean part anticommutes with both γ_0 and γ_5, giving the properties

$$\gamma_5 D_c = D_c^\dagger \gamma_5, \qquad \gamma_0 D_c = D_c^\dagger \gamma_0. \tag{3}$$

The eigenvalues of D_c lie along the line $\mathrm{Re}\lambda = m$ and occur in complex conjugate pairs; i.e. if we have $D_c\chi = \lambda\chi$, then $D_c\gamma_5\chi = \lambda^*\gamma_5\chi$.

The free theory is particularly simple in momentum space, with $D_c = i\vec{p}\cdot\vec{\gamma} + m$. The eigenvalues are $\lambda = \pm i|\vec{p}| + m$. This spectrum is discrete in finite volume, with momentum quantized in units of $\frac{2\pi}{L}$, where L is the size of our box.

Now for the lattice, "naive" fermions replace the momentum with a trigonometric function $p_i \to \sin(p_i a)/a$. This causes the well known doubling, with extra low energy states when $p_i \sim \frac{\pi}{a}$. The Wilson solution to this problem

*This manuscript has been authored under contract number DE-AC02-98CH10886 with the U.S. Department of Energy. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.

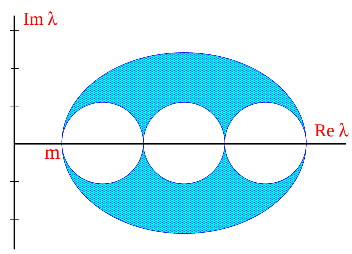
†Supported in part by DOE grant DE-FG05-84ER40154.

‡Supported in part by DOE grant DE-FG02-96ER40949 and by a Guggenheim fellowship. H. N. would like to thank L. Baulieu and the entire group at LPTHE for their hospitality and support.

gives the mass a momentum dependence $m \to m + \frac{1}{a}\sum_i (1-\cos(p_i a))$ so that the doubler mass becomes of $O(1/a)$. The free Wilson-Dirac operator in the Hamiltonian formulation is thus

$$D_w = m + \frac{1}{a}\sum_i \left(i\sin(p_i a)\gamma_i + 1 - \cos(p_i a)\right) \quad (4)$$

The eigenvalues of D_w lie on a superposition of ellipses, as sketched here



To simplify notation, from now on we work in lattice units with $a = 1$. For the continuum limit we are interested in small eigenvalues $\lambda \sim 0$.

When gauge fields are added, several things happen. First the fermi eigenvalues move around, starting to fill the holes in the above figure. In the process the operator D_w ceases to be normal, i.e $[D_w, D_w^\dagger] \neq 0$. The eigenvectors are no longer orthogonal. Also the pairing due to the γ_5 symmetry is lost. This is directly related to the chiral symmetry violation in the Wilson term.

To get a nicer behavior, we mimic ref. [4] and project D_w onto a unitary matrix

$$V = D_w (D_w^\dagger D_w)^{-1/2} \quad (5)$$

Being unitary, $V^\dagger V = 1$, this is a normal matrix. From this we construct the operator

$$D = 1 + V \quad (6)$$

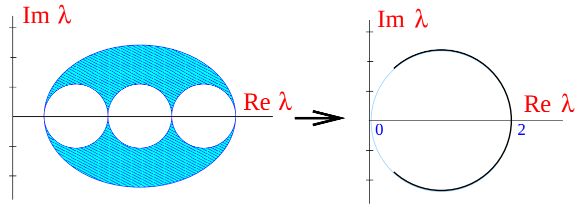
and our new fermion Hamiltonian

$$\begin{aligned} H_f &= \psi^\dagger h \psi \\ h &= \gamma_0 D = \gamma_0(1+V) \end{aligned} \quad (7)$$

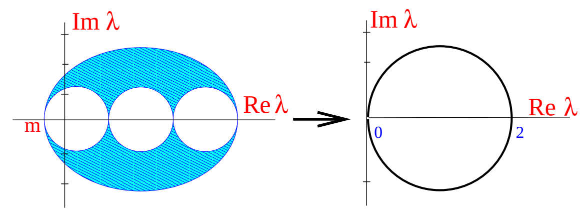
We recover the continuum properties

$$[D, D^\dagger] = 0, \qquad \gamma_5 D = D^\dagger \gamma_5 \quad (8)$$

and the eigenvalues again lie in complex conjugate pairs. By construction, the eigenvalues lie on a circle, as sketched here



As the above figure is drawn, there are no low energy states. For such, we must make the starting Wilson mass m negative, keeping the doubler masses positive. Then the figure is more like this



As with the euclidian overlap operator, combining the unitarity condition $V^\dagger V = 1 = D^\dagger D - D - D^\dagger + 1$ with the hermiticity condition $D^\dagger = \gamma_5 D \gamma_5$ gives rise to the Ginsparg-Wilson [5] relation

$$\gamma_5 D + D\gamma_5 - D\gamma_5 D = 0 \quad (9)$$

However we now have another variation on this following from $D^\dagger = \gamma_0 D \gamma_0$

$$\gamma_0 D + D\gamma_0 - D\gamma_0 D = 0 \quad (10)$$

By either multiplying the first of these two Ginsparg-Wilson relations by γ_0 or the second by γ_5 we obtain the exact commutation relation

$$\left[\gamma_5(1 - \frac{D}{2}), h\right] = 0 \quad (11)$$

This suggests defining an axial charge

$$\begin{aligned} Q_5 &\equiv \psi^\dagger q_5 \psi \\ q_5 &\equiv \gamma_5(1 - \tfrac{D}{2}) = \gamma_5 \tfrac{1-V}{2} \end{aligned} \quad (12)$$

This charge commutes with the fermion part of our Hamiltonian (see also Ref. [6])

$$[Q_5, H_f] = 0. \quad (13)$$

Since D can be reconstructed from either h or q_5, the matrices q_5 and h are closely correlated

$$\begin{aligned} D &= \gamma_0 h = 2 - 2\gamma_5 q_5 \\ \gamma_5 q_5 &+ \gamma_0 \tfrac{h}{2} = 1 \end{aligned} \quad (14)$$

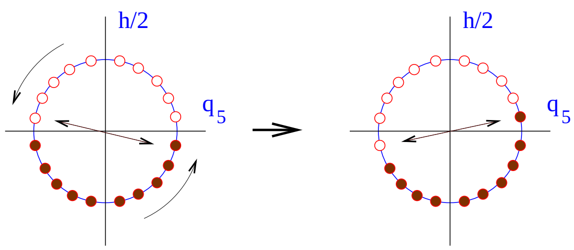
Squaring this gives

$$q_5^2 + \frac{h^2}{4} = 1. \qquad (15)$$

Since q_5 and h commute they can be simultaneously diagonalized. The above equation states that these eigenvalues lie on a circle.

Low-energy states have a well defined chirality; i.e. $h \sim 0 \Rightarrow q_5 \sim \pm 1$. In contrast, high-energy states all have $|q_5| < 1$. The combination $\gamma_0\gamma_5$ flips the sign of both eigenvalues, which are thus paired on opposite sides of the the circle.

This formulation gives a simple understanding of anomalies, analogous to the domain-wall discussion in [7]. An adiabatic change of gauge fields shifts eigenvalues continuously while the pairing is preserved. A topologically non-trivial shift moves an $h > 0$ eigenvalue to $h < 0$ while its paired state goes the other way. In this way a "left" hole and "right" particle (or vice versa) are generated out of the Dirac sea, as sketched here



We have been considering the fermion states in a given background gauge field. For the full coupled quantum theory theory we add the gauge field Hamiltonian

$$H = H_f + E^2 + B^2 \qquad (16)$$

The operator D involves link variables, which do not commute with the electric field term, $\left[Q_5, E^2\right] \neq 0$. This is essential for the $U(1)$ anomaly to generate the η' mass via mixing with gluons.

There is a close connection between these zero crossings in energy and zero modes in Euclidean space. Adiabatically change the gauge field with time, $\frac{d}{dt}h(t) = O(1/T)$ with $-\frac{T}{2} < t < \frac{T}{2}$. Then consider an eigenvalue $h(t)\chi(t) = E_i(t)\chi(t)$ which changes in sign, $E(-\frac{T}{2}) < 0$, $E(\frac{T}{2}) > 0$. From this construct

$$\phi(t) = e^{-E(t)t}\chi(t) \qquad (17)$$

This satisfies

$$D_4\phi \equiv \gamma_0 \left(\partial_0 + h(t)\right) \phi(t) = 0 + O(1/T) \qquad (18)$$

This wave function is normalizable if the eigenvalue rises through 0.

Because of γ_5 hermiticity, $\gamma_5 D_4 \gamma_5 = D_4^\dagger$, complex eigenvalues of D_4 are paired; thus, unpaired zero modes are robust. This is the lattice version of the index theorem.

A variety of questions remain. One involves flavored axial charges such as $Q_5^\alpha = \psi^\dagger \tau^\alpha \gamma_5 (1 - \frac{D}{2})\psi$. It appears that these also do not commute with the electric field term of the Hamiltonian, $\left[Q_5^\alpha, E^2\right] \neq 0$. Why is this so, despite the euclidean overlap formulation having an exact flavored chiral symmetry? Another question arises at the level of currents. Since Q_5 is not ultra locally defined [8], what is the natural associated $\vec{J}_5$? Finally, our construction was for vectorlike theories. Can something similar be done for chiral gauge theories? In these cases anomalies change species as in the t'Hooft [9] baryon decay process. For the standard model, how do the quark and lepton "circles" interact to give this phenomenon?

REFERENCES

1. M. Creutz, Rev. Mod. Phys. 73 (2001) 119.
2. D. Kaplan, Phys. Lett. B288 (1992) 342.
3. R. Narayanan and H. Neuberger, Phys. Lett. B302 (1993) 62; Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. B412 (1994) 574; Nucl. Phys. B443 1995 305.
4. H. Neuberger, 1998b Phys. Lett. B417 (1998) 141; H. Neuberger, Phys. Lett. B427 (1998) 353.
5. P. H. Ginsparg and K. G. Wilson, Phys. Rev. D 25 (1982) 2649.
6. I. Horváth and H.B. Thacker, Nucl. Phys. Proc. Suppl. **73** (1999) 682.
7. M. Creutz and I. Horváth, Phys. Rev. D 50 (1994) 2297.
8. I. Horváth, Phys. Rev. Lett. **81** (1998) 4063.
9. G. 't Hooft, Phys. Rev. Lett. 37 (1976) 8; Phys. Rev. D14 (1976) 3432.