

Transiting topological sectors with the overlap

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The overlap operator provides an elegant definition for the winding number of lattice gauge field configurations. Only for a set of configurations of measure zero is this procedure undefined. Without restrictions on the lattice fields, however, the space of gauge fields is simply connected. I present a simple low dimensional illustration of how the eigenvalues of a truncated overlap operator flow as one travels between different topological sectors.

The overlap operator [1] elegantly extends many features of chiral symmetry to the lattice. In particular, it provides a precise definition of a “winding number” for gauge field configurations, giving a lattice extension of continuum index theorems [2]. At first sight this seems remarkable since the space of Wilson gauge fields is simply connected. In selecting sectors, the overlap operator must become singular at boundaries. These singular configurations form a set of measure zero. A simple “admissibility” criterion [3] guarantees that the overlap operator is well defined. This criterion, however, is rather strong, and is not generally satisfied for configurations in practical simulations.

Here I explore the behavior of the overlap operator as one passes through a singularity separating two different sectors. This requires a truncation of the definition of the overlap. The result is that two complex eigenvalues of the overlap operator collide and evolve into a zero mode plus one heavy real eigenvalue. I follow this evolution explicitly in a simple zero space-time dimensional toy model.

I briefly review the so called “continuum” Dirac action and the role of zero modes. The generic action for a gauge theory consists of a pure gauge term and an interaction with the fermions, $S = S_g + S_f$. The gauge part is the square of the field strength, $S_g = \frac{1}{4} \int F_{\mu\nu} F_{\mu\nu}$. The fermion term is a quadratic form $S_f = \int \bar{\psi} D_c \psi$ with

$$D_c = \gamma \cdot (\partial + igA) + m \quad (1)$$

The differential operator D_c consists of an anti-Hermitian kinetic term plus a Hermitian mass term. It satisfies the Hermiticity condition

$$\gamma_5 D_c = D_c^\dagger \gamma_5 \quad (2)$$

Complex eigenvalues of D_c are paired; if $D_c \chi = \lambda \chi$ then $D_c \gamma_5 \chi = \lambda^* \gamma_5 \chi$. Restricting ourselves to the space spanned by eigenvectors with real eigenvalues, then γ_5 and D_c can be simultaneously diagonalized. On this subspace, an integer index is the difference of the number of positive and negative eigenvalues of γ_5 , i.e. $\nu = n_+ - n_-$. This number is robust under smooth field deformations and lies at the basis of the index theorem, which says that this index can also be calculated directly from the gauge fields as a topological charge [2].

For comparison with the lattice theory, we can consider the free continuum theory in momentum space $D_c = ip \cdot \gamma + m$. This has eigenvalues $\lambda = \pm i|p| + m$. If we work in finite volume, the momentum is quantized in units of $\frac{2\pi}{L}$. So called “naive” lattice fermions are obtained from the continuum result by the simple substitution $p_\mu \rightarrow \sin(p_\mu a)/a$. These are fraught with the

*This manuscript has been authored under contract number DE-AC02-98CH10886 with the U.S. Department of Energy. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes. I benefited from discussions with Y. Shamir and M. Golterman during visits made possible by grant No. 98-302 from the United-States – Israel Binational Science foundation (BSF), Jerusalem, Israel.

famous doublers, extra low energy states whenever any component of the momentum satisfies $p_\mu \sim \frac{\pi}{a}$. The doubling problem was solved years ago by Wilson [4], who allowed the fermion mass to depend on momentum

$$m \rightarrow m + \frac{1}{a} \sum_{\mu} (1 - \cos(p_\mu a)) \quad (3)$$

thus giving the doublers a mass of order $1/a$. In momentum space, the free Wilson-Dirac operator takes the form

$$D_w = m + \frac{1}{a} \sum_{\mu} (i \sin(p_\mu a) \gamma_\mu + 1 - \cos(p_\mu a)). \quad (4)$$

The difficulty with the Wilson approach is that the added term violates chiral symmetry. With gauge fields, the eigenvalues drift. To maintain the physics of massless quarks requires fine tuning. Real eigenvalues of D_w can appear along much of the real axis, and for the purpose of defining an index we need a criterion for which of them to include.

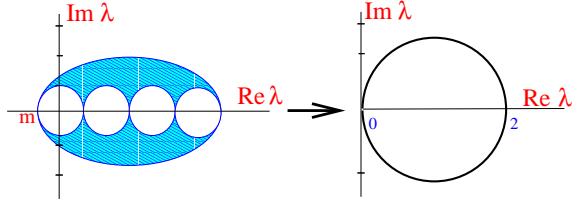
The overlap Dirac operator partially answers these questions [1]. To construct this operator, one starts with D_w at a negative m . This is projected onto a unitary matrix

$$V = D_w (D_w^\dagger D_w)^{-1/2} \quad (5)$$

from this the overlap operator is simply

$$D = 1 + V \quad (6)$$

Thus the low eigenvalues of D_w become low eigenvalues of D , while the higher ones are projected to the side of the unitarity circle near unity. This process is sketched here



This construction satisfies the continuum property that D is normal, $[D, D^\dagger] = 0$, and preserves the γ_5 Hermiticity, $\gamma_5 D = D^\dagger \gamma_5$. Furthermore, we have the famous Ginsparg-Wilson relation [5], succinctly written as

$$D \gamma_5 = -\Gamma_5 D \quad (7)$$

with the new matrix

$$\Gamma_5 \equiv V \gamma_5 = (1 - D) \gamma_5 \quad (8)$$

satisfying some of the same conditions as γ_5 , $\Gamma_5 = \Gamma_5^\dagger$ and $\Gamma_5^2 = \gamma_5^2 = I$. The eigenvalues of Γ_5 are all ± 1 , implying its trace is an integer. From this we define the gauge field index

$$\nu = \frac{1}{2} \text{Tr} \Gamma_5 \quad (9)$$

Note the factor of 2, which comes from heavy modes at $V \sim 1$.

Our fermionic action, $S_f = \bar{\psi} D \psi$, is invariant under the generalized chiral rotation

$$\psi \rightarrow e^{i\theta \gamma_5} \psi \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\theta \Gamma_5} \quad (10)$$

In this formalism the chiral anomaly appears in the fermionic measure

$$d\bar{\psi} d\psi \rightarrow d\bar{\psi} d\psi e^{-i \text{Tr} \Gamma_5} \rightarrow d\bar{\psi} d\psi e^{-2i\theta \nu} \quad (11)$$

much as in continuum discussions [6].

A fermion mass introduces, as in the continuum, the possibility of a CP violating term. To have formulas similar to the continuum, it is convenient to consider the fermionic action with mass term of the form

$$S_f = \bar{\psi} D \psi + \bar{\psi} (1 - V) M \psi / 2 \quad (12)$$

The rotation $M \rightarrow e^{i\theta \gamma_5} M$ is physically equivalent to a modification of the gauge action $S_g \rightarrow S_g + \theta \nu$. This angle θ is the strong CP violating parameter of innumerable continuum discussions.

This is all quite elegant, but the space of Wilson lattice gauge fields is simply connected. This raises the question of what happens as one continues between topological sectors. Along such a path D must become singular. To keep things well defined, I introduce a cutoff into the definition of V

$$V = D_w (D_w^\dagger D_w + \epsilon^2)^{-1/2} \quad (13)$$

The quantity ϵ should be analogous to $\frac{1}{L_5}$ with domain wall fermions.

I now introduce a simple 2×2 matrix example. I take effectively zero space-time dimensions, with σ_3 playing the role of γ_5 . The hermiticity condition reduces to $D_W^\dagger = \sigma_3 D_W \sigma_3$. The most general two by two matrix satisfying this has the form

$$D_W = b_0 + i b_1 \sigma_1 + i b_2 \sigma_2 + b_3 \sigma_3 \quad (14)$$

This is singular when $|D_W| = b_0^2 + b_1^2 + b_2^2 - b_3^2 = 0$. We have a Minkowski space with the role of time being played by b_3 . Minkowski space naturally breaks into light-like and space-like sectors. The index will highlight this division.

It is convenient to go to an analogue of “polar” coordinates and reparametrize

$$D_W = U (a_0 + a_3 \sigma_3) U \quad (15)$$

with $U = e^{i(a_1 \sigma_1 + a_2 \sigma_2)/2}$. The coordinate mapping is $b_3 = a_3$ and $a_0 = \pm \sqrt{b_0^2 + b_1^2 + b_2^2}$. We explicitly construct V

$$V = D_W (D_W^\dagger D_W + \epsilon^2)^{-1/2} = U \begin{pmatrix} \frac{a_0 + a_3}{\sqrt{(a_0 + a_3)^2 + \epsilon^2}} & 0 \\ 0 & \frac{a_0 - a_3}{\sqrt{(a_0 - a_3)^2 + \epsilon^2}} \end{pmatrix} U \quad (16)$$

The possible topological sectors fall into three cases. The first has $a_0^2 - a_3^2 > 0$ representing the spacelike sector of our Minkowski space ($a_3 = b_3$ plays the role of time). In this case $V = U^2$ and thus $D = 1 + U^2$. This has a conjugate pair of eigenvalues $\lambda_{\pm} = 1 + e^{\pm i \sqrt{a_1^2 + a_2^2}}$. The winding number vanishes: $\nu = \frac{1}{2} \text{Tr } U \gamma_5 U^\dagger = 0$

The second case involves $a_3 > |a_0|$. Then $V = \sigma_3$ and $D = 1 + \sigma_3$. The winding number $\nu = 1$; so, this represents the analog of an “instanton”. The third and final case is a reflection of this, with $a_3 < -|a_0|$, $D = 1 - \sigma_3$, and $\nu = -1$.

Now I transit between these sectors. As an example, let a_3 pass through the “light cone” at $a_0 > 0$. To be explicit, use $U = c - i s \sigma_2$ with $c^2 + s^2 = 1$. For our interpolation parameter, define $x = a_0 - a_3 / \sqrt{(a_0 - a_3)^2 + \epsilon^2}$ with range $-1 \leq x \leq 1$. With the cutoff in place V is no longer unitary, but takes the form

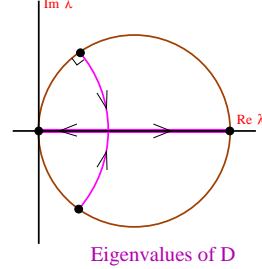
$$V = \frac{1}{2} \begin{pmatrix} 1 + c - x + cx & -s - sx \\ s + sx & -1 + c + x + cx \end{pmatrix} \quad (17)$$

The eigenvalues of this are

$$\lambda = \frac{1}{2} \left(c(1+x) \pm \sqrt{c^2(1+x)^2 - 4x} \right) \quad (18)$$

As an eigenvalue of $D_W^\dagger D_W$ passes through zero, a pair of eigenvalues leaves the unitarity circle. This is a perpendicular departure, following another circle. These eigenvalues then collide and become real. They move out to rest at

± 1 . In the process the winding number changes by one unit. This behavior is sketched here



The participating eigenvalues can come from anywhere on the unitarity circle. An instanton falling through the lattice does not require a large fermionic action. Throughout this discussion the index ν is an integer except within ϵ of sector boundaries. This behavior is fairly robust, with other eigenvalues of V moving little. However, as shown in [7] the other eigenvalues can also briefly leave the unitarity circle as we pass through the boundary.

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