Effective potentials, thermodynamics, and twisted mass quarks

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Using effective Lagrangian arguments, I explore the qualitative behavior expected at finite temperature for two-flavor lattice QCD formulated with Wilson fermions and a twisted mass term. A rather rich phase structure is predicted, exhibiting Aoki's parity violating phase along with a deconfinement region forming a conical structure in the space of coupling, hopping parameter, and twisted mass variables.

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I. INTRODUCTION

Nonperturbative phenomena are crucial to understanding strong interaction physics. Two particularly powerful tools in this context are lattice gauge theory and effective field theories. Combining these approaches can frequently give new insights. For example, effective Lagrangians can model the mechanisms for the lattice artifacts that mutilate chiral symmetry, explaining [1,2] such phenomena as the spontaneous breaking of parity and charge conjugation discussed long ago by Aoki [3]. Attempting to reduce artifacts stimulates the development of new lattice actions, such as domain wall [4,5] or overlap [6,7] fermions. The chiral symmetry inherent in effective Lagrangian approaches also has been crucial in exposing the issues inherent in the rooting procedure often used to adjust the number of quark species [8].

One of the major successes of lattice QCD is the quantitative estimate of the temperature for the transition from ordinary hadronic matter to a plasma of quarks and gluons [9]. Chiral symmetry has an interesting interplay with this transition; indeed, it appears that this symmetry is restored at the same temperature as deconfinement. Thus, finite temperature QCD is a natural playground for gaining a better understanding of chiral symmetry.

This paper brings together the three topics of lattice artifacts, chiral symmetry, and the deconfinement transition. The presence of lattice artifacts brings in new types of mass terms, one of which is often referred to as a "twisted" mass [10]. The main predictions are a rather intricate phase structure which can be looked for in numerical simulations. The phase structure with twisted mass at zero temperature has been described in Refs. [11–13]. The main addition here is the inclusion of the deconfinement transition in this picture.

One might well ask why care about lattice artifacts? Of course this is necessary to understand the limitations of lattice simulations. At a more practical level, it is through understanding these effects that one can explore the reasons for the failure of the rooting procedure used with staggered quarks. At the conceptual level, seeing the distortions of chiral symmetry can help one understand the nature of this symmetry and how chiral anomalies work on the lattice. These issues are closely related to how quark masses are defined [14]. One might also hope that understanding these aspects can give insight into why gauge theories coupled to chiral currents, such as in the standard model, are so hard to put on the lattice. And finally, we will see that the lattice artifacts give rise to a rather amusing and somewhat complicated phase structure.

For simplicity, this discussion is restricted to QCD with two degenerate quark flavors. With three or more flavors the mass "twisting" is more complicated and not unique. Furthermore, we only consider the theory without any CP noninvariant term associated with the gauge field topology.

Section II starts off with an extremely brief review of the Wilson fermion formulation [15]. The discussion then turns in Sec. III to the parameters of QCD, emphasizing the nonlinear mapping between the physical continuum parameters and the bare parameters used on the lattice. Then Sec. IV reviews the lattice artifacts introduced through the chiral symmetry breaking properties of the Wilson fermion action. Section V introduces the twisted mass term, which only has meaning in the context of the lattice artifacts. Section VI brings finite temperature into the picture, showing how deconfinement at high temperature interplays with the previous lattice artifacts. Section VII combines the previous ideas to predict qualitatively the rich phase structure that should appear in the space of lattice parameters. In particular, the deconfined phase should appear as an approximately conical structure which should be looked for in simulations. Finally remarks on some open questions appear in Sec. VIII.

II. WILSON FERMIONS

For completeness this section gives a brief discussion of the Wilson fermion approach. The motivation is to overcome the doubling issue of naive fermions by adding a momentum-dependent mass to the extra states. Doubling arises because on the lattice physics becomes naturally periodic in momenta, with momentum components being replaced by trigonometric functions, i.e. $p_\mu \rightarrow \frac{1}{a}\sin(p_\mu a)$. The problem is that this quantity is small not just for small momentum, but also for $p_\mu \sim \pi/a$. Adding a further

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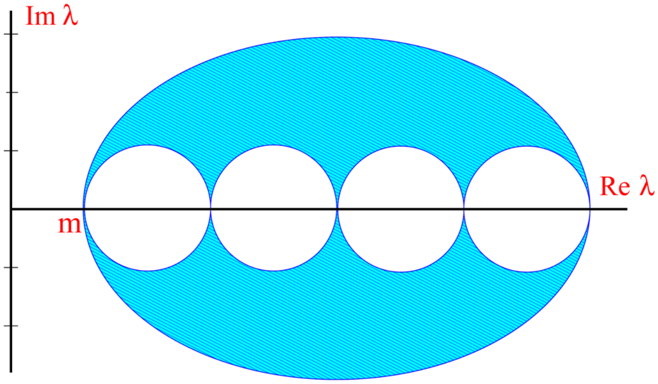

FIG. 1 (color online). The eigenvalue spectrum of the free Wilson fermion operator is a set of nested circles. On turning on the gauge fields, some eigenvalues drift into the open regions. Some complex pairs can collide and become real. These are connected to gauge field topology.

trigonometric behavior in the form of a momentum-dependent mass, the extraneous doublers can be made heavy. For the free case, the simplest version of Wilson fermions uses the Dirac operator in momentum space

$$D_W(p) = \frac{1}{a} + \frac{2K}{a}\sum_\mu [i\gamma_\mu \sin(p_\mu a) - \cos(p_\mu a)]. \quad (1)$$

The physical fermion mass is read off from the small momentum behavior as $m = \frac{1}{2a}(1/K - 8)$. This vanishes at $K = K_c = 1/8$. The eigenvalues of this free operator lie on a set of "nested circles," as sketched in Fig. 1.

This discussion assumes that the gauge fields are formulated as usual with group valued matrices on the lattice links. To be specific, one can consider using the simple Wilson gauge action as a sum over plaquettes

$$S_g = \frac{\beta}{3}\sum_p \mathrm{Re\,Tr} U_p, \quad (2)$$

although this specific form is not essential the qualitative nature of the phase diagram. When the gauge fields are turned on, the dynamics will move the fermion eigenvalues around, partially filling the holes in eigenvalue pattern of Fig. 1. Some eigenvalues can become real and are related to gauge field topology [16].

Note that the basic lattice theory has two parameters, β and K. These are related to bare coupling, $\beta \sim 6/g_0^2$, and quark mass, $(1/K - 1/K_c) \sim m_q$. This will be discussed further in the next section.

III. LATTICE VERSUS CONTINUUM PARAMETERS

The quark confining dynamics of the strong interactions, QCD, is a remarkably economical theory in terms of the number of adjustable parameters. First of these is the overall strong interaction scale, $\Lambda_{\rm qcd}$. This is scheme dependent, but once a renormalization procedure has been selected, it is well defined. It is not independent of the coupling constant, the connection being fixed by asymptotic freedom. In addition, the theory can depend on the quark mass, or more precisely the dimensionless ratio $m/\Lambda_{\rm qcd}$. As with the overall scale, the definition of m is scheme dependent. The two-flavor theory with degenerate quarks has only one such mass parameter. On considering the theory at a finite physical temperature, this adds another parameter, which can also be considered in units of the overall scale, i.e. consider the ratio $\frac{T}{\Lambda_{\rm qcd}}$. Finally, to relate things to the theory with a lattice cutoff, add a scale for this cutoff. As with everything else, measure this in units of the overall scale; so, the fourth parameter is $a\Lambda_{\rm qcd}$, where one can regard a as the lattice spacing.

The goal here is to explore the lattice artifacts that arise with Wilson fermions [15]. On the lattice it is generally easier to work directly with lattice parameters. One of these is the plaquette coupling β, which, with the usual conventions, is related to the bare coupling $\beta = 6/g_0^2$. For the quarks, the natural lattice quantity is the "hopping parameter" K. To include finite temperature effects, it is customary to work with a finite temporal lattice of N_t sites. And finally, the connection with physical scales appears via the lattice spacing a.

The set of physical parameters and the set of lattice parameters are, of course, equivalent, and there is a well-understood nonlinear mapping between them

$$\left\{\frac{m}{\Lambda_{\rm qcd}}, \frac{T}{\Lambda_{\rm qcd}}, a\Lambda_{\rm qcd}\right\} \leftrightarrow \{\beta, K, N_t\}. \quad (3)$$

Of course, to extract physical predictions we are interested in the continuum limit $a\Lambda_{\rm qcd} \rightarrow 0$. For this, asymptotic freedom tells us we must take $\beta \rightarrow \infty$ at a rate tied to

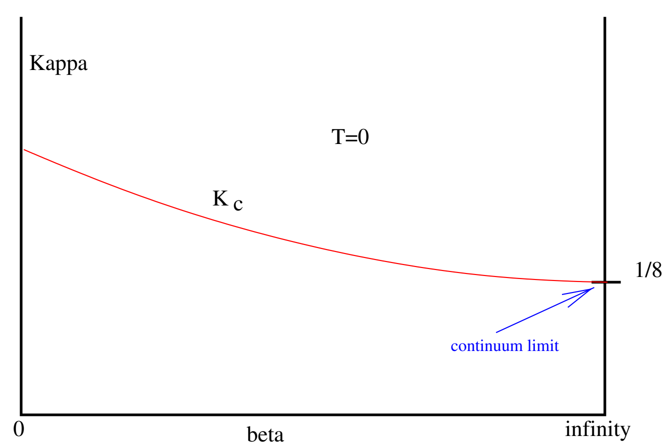

FIG. 2 (color online). The continuum limit of lattice gauge theory with Wilson fermions occurs at $\beta \rightarrow \infty$ and $K \rightarrow 1/8$. Coming in from this point to finite beta is the curve $K_c(\beta)$, representing the lowest phase transition in K for fixed beta. The nature of this phase transition is a delicate matter, discussed in the text.

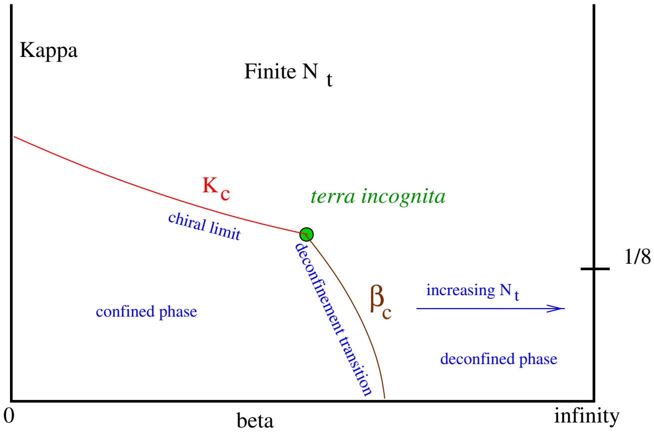

FIG. 3 (color online). At finite N_t the deconfinement phase transition appears at a finite value for β which depends on the hopping parameter K. For the continuum limit N_t is taken to infinity and this structure moves towards $\beta = \infty$. The region above $K_c(\beta)$ is the object of later discussion.

$\Lambda_{\rm qcd}$. Simultaneously we must take the hopping parameter to a critical value. With normal conventions, this takes $K \to K_c \to 1/8$ at a rate tied to desired quark mass m. Finally, the number of temporal sites must also go to infinity as $N_t = \frac{1}{aT}$. Figure 2 sketches how the continuum limit is taken in the β, K plane for zero temperature. Figure 3 sketches how the basic phase diagram is modified for a finite number of time slices. Some early results on this structure from numerical simulation appear in Ref. [17]. The subject of this paper is to further explore this phase diagram with particular attention to hopping parameters larger than K_c. This discussion adds the possible twisted mass term (defined later) to the exposition in [1]. Some very preliminary studies of this system at finite temperature are presented in Ref. [18].

IV. EFFECTIVE CHIRAL LAGRANGIANS AND LATTICE ARTIFACTS

The use of effective field theories to describe the interactions of the pseudoscalar mesons is an old and venerable topic. Here we will only discuss the simplest form for the two-flavor theory, adding terms that mimic the expected lattice artifacts. The language is framed in terms of the isovector pion field $\vec{\pi} \sim i\bar{\psi}\gamma_5\vec{\tau}\psi$ and the scalar sigma $\sigma \sim \bar{\psi}\psi$. The starting point for this discussion is the canonical "Mexican hat" potential

$$V_0 = \lambda(\sigma^2 + \vec{\pi}^2 - v^2)^2 \quad (4)$$

schematically sketched in Fig. 4. The potential has a symmetry under $O(4)$ rotations amongst the pion and sigma fields expressed as the four-vector $\Sigma = (\sigma, \vec{\pi})$. This represents the axial symmetry of the underlying quark theory.

As usually considered, this theory is taken with the minimum for the potential occurring at a nonvanishing

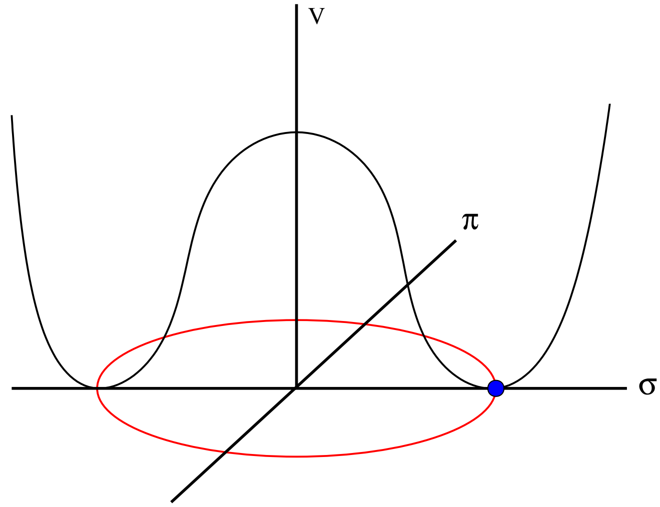

FIG. 4 (color online). The symmetric "Mexican hat" potential for the pion and sigma fields before mass terms and lattice artifacts are turned on.

value for the fields. This is a classic example of spontaneous symmetry breaking, and the vacuum is conventionally selected to lie in the sigma direction with $\langle\sigma\rangle > 0$. The pions are then Goldstone bosons of the theory, being massless because the potential provides no barrier to oscillations of the fields in the pionic directions.

With this potential, it is natural to include a quark mass by adding a constant times the sigma field

$$V_1 = -m\sigma. \quad (5)$$

This explicitly breaks the chiral symmetry by "tilting" the potential as sketched in Fig. 5. This selects a unique vacuum which, for $m > 0$, gives a positive expectation for sigma. In the process the pions gain a mass, with $m_\pi^2 \sim m$.

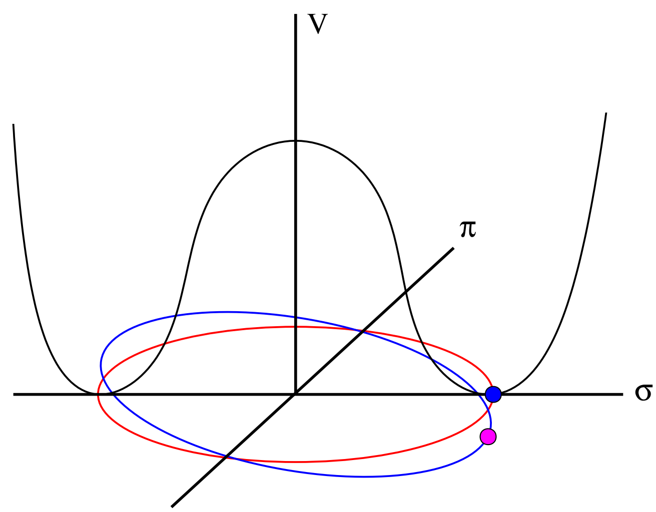

FIG. 5 (color online). A mass terms acts by tilting the potential, thus selecting a unique vacuum.

There is an obvious freedom in the selection of the mass term. Because of the symmetry of V_0, it does not physically matter in which direction we tilt the vacuum. In particular, a mass term of form

$$m\sigma \rightarrow m\cos(\theta)\sigma + m\sin(\theta)\pi_3 \tag{6}$$

should give equivalent physics for any θ. However, as we will see, lattice artifacts can break this symmetry, introducing the possibility of physics at finite lattice spacing which depends on this angle. The second term in this equation is what we will call the "twisted mass."

The Wilson term inherently breaks chiral symmetry. This will give rise to various modifications of the effective potential. The first correction is expected to be an additive contribution to the quark mass, i.e. an additional tilt to the potential. This means that the critical kappa, defined as the smallest kappa where a singularity is found in the β, K plane, will move away from the limiting value of $1/8$. Thus we introduce the function $K_c(\beta)$ and imagine that the mass term is modeled with the form

$$m \rightarrow c_1(1/K - 1/K_c(\beta)). \tag{7}$$

In general the modification of the effective potential will have higher order corrections. A natural way to include such is as an expansion in the chiral fields. With this motivation we include a term in the potential of form

$$c_2\sigma^2. \tag{8}$$

Such a term was considered in [1,2]. The predicted phase structure depends qualitatively on the sign of c_2, but *a priori* we have no information on this. Indeed, as it is a lattice artifact, it is expected that this sign might depend on the choice of gauge action. Note that we could have added a term like $\vec{\pi}^2$, but this is essentially equivalent since $\vec{\pi}^2 = (\sigma^2 + \vec{\pi}^2) - \sigma^2$, and the first term here can be absorbed, up to an irrelevant constant, into the starting Mexican hat potential.

First consider the case when c_2 is less than zero, thus lowering the potential energy when the field points in the positive or negative sigma direction. This quadratic warping helps to stabilize the sigma direction, as sketched in Fig. 6, and the pions cease to be true Goldstone bosons when the quark mass vanishes. Instead, as the mass passes through zero, we have a first-order transition as the expectation of σ jumps from positive to negative. This jump occurs without any physical particles becoming massless.

Things get a bit more complicated if $c_2 > 0$, as sketched in Fig. 7. In that case the chiral transition splits into 2 second-order transitions separated by phase with an expectation for the pion field, i.e. $\langle\vec{\pi}\rangle \neq 0$. Since the pion field has odd parity and charge conjugation as well as carries isospin, all of these symmetries are spontaneously broken in the intermediate phase. As isospin is a continuous group, this phase will exhibit Goldstone bosons. The number of these is two, representing the two flavored generators orthogonal to the direction of the expectation value. If higher order terms do not change the order of the transitions, there will be a third massless particle exactly at the transition endpoints. In this way the theory does acquire three massless pions exactly at the transitions, as discussed by Aoki [3]. The intermediate phase is usually referred to as the "Aoki phase."

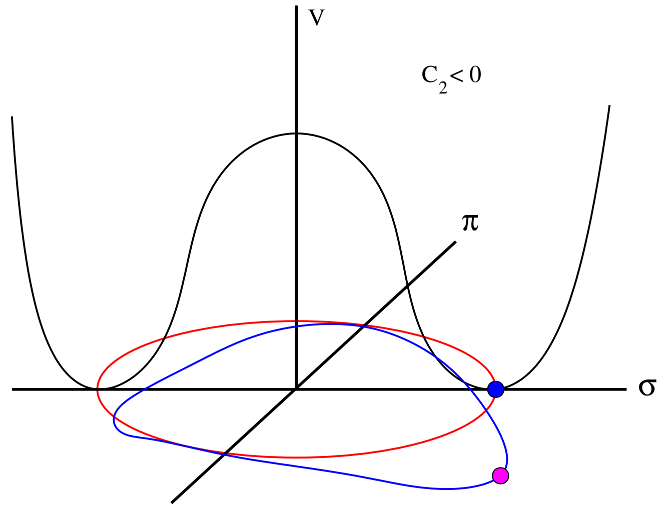

FIG. 6 (color online). Lattice artifacts could quadratically warp the effective potential. If this warping is downward in the sigma direction, the chiral transition becomes first order without the pions becoming massless.

To include these ideas in the effective model, add the c_2 term to the potential

$$V(\vec{\pi}, \sigma, L) = \lambda(\sigma^2 + \vec{\pi}^2 - v^2)^2 - c_1(1/K - 1/K_c(\beta))\sigma + c_2\sigma^2. \tag{9}$$

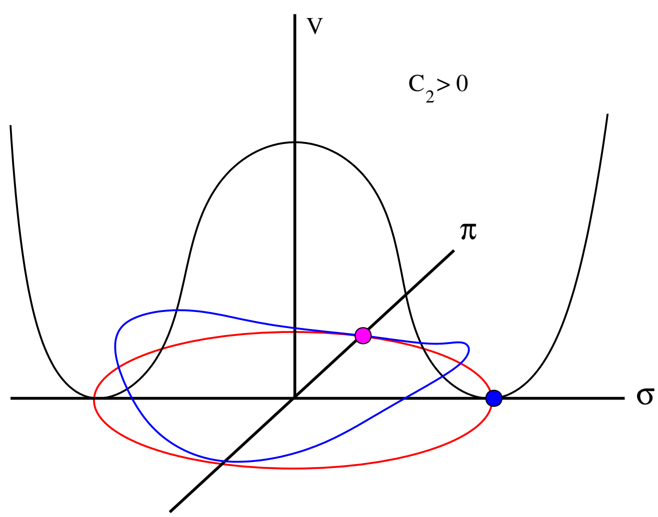

FIG. 7 (color online). If the lattice artifacts warp the potential upward in the sigma direction, the chiral transition splits into 2 second-order transitions separated by a phase where the pion field has an expectation value.

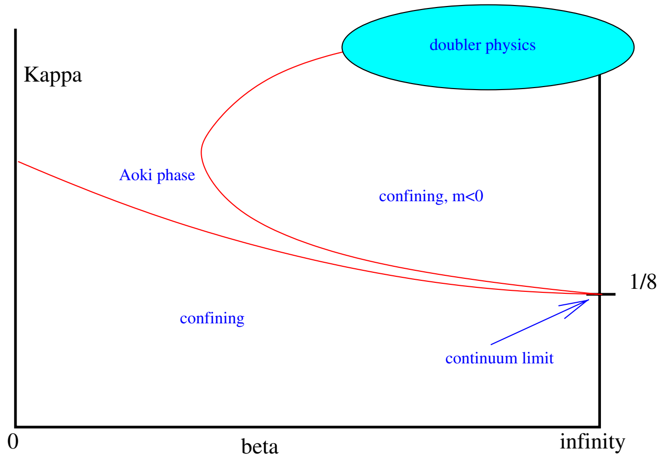

FIG. 8 (color online). The qualitative structure of the β, K plane including the possibility of an Aoki phase.

Assuming the $c_2 > 0$ case, Fig. 8 shows the qualitative phase diagram expected.

V. TWISTED MASS

The c_2 term breaks the equivalence of different chiral directions. This means that physics will indeed depend on the angle θ if one takes a mass term of the form in Eq. (6). Consider complexifying the fermion mass in the usual way

$$m\bar{\psi}\psi \longrightarrow \frac{1}{2}(m\bar{\psi}_L\psi_R + m^*\bar{\psi}_R\psi_L). \tag{10}$$

The rotation of Eq. (6) is equivalent to giving the up and down quark masses opposite phases

$$m_u \longrightarrow e^{+i\theta}m_u \tag{11}$$

$$m_d \longrightarrow e^{-i\theta}m_d. \tag{12}$$

Thus motivated, we can consider adding a new mass term to the lattice theory

$$\mu i\bar{\psi}\tau_3\gamma_5\psi \sim \mu\pi_3. \tag{13}$$

This extends our effective potential to

$$V(\vec{\pi}, \sigma) = \lambda(\sigma^2 + \vec{\pi}^2 - v^2)^2 - c_1(1/K - 1/K_c(\beta))\sigma + c_2\sigma^2 - \mu\pi_3. \tag{14}$$

The twisted mass represents the addition of a "magnetic field" conjugate to the order parameter for the Aoki phase.

There are a variety of motivations for adding such a term to our lattice action [10,13]. Primary among them is that $O(a)$ lattice artifacts can be arranged to cancel. With two flavors of conventional Wilson fermions, these effects change sign on going from positive to negative mass, and if we put all the mass into the twisted term we are halfway between. It should be noted that this cancellation only occurs when all the mass comes from the twisted term; for other combinations with a traditional mass term, some lattice artifacts of $O(a)$ will survive. Also, although it looks like we are putting phases into the quark masses, these cancel between the two flavors, and the resulting fermion determinant remains positive. Furthermore, the algorithm is considerably simpler and faster than either overlap [6,7] or domain wall [4,5] fermions while avoiding the diseases of staggered quarks [8]. Another nice feature of adding a twisted mass term is that it allows a better understanding of the Aoki phase and shows how to continue around it. Figures 9 and 10 show how this works for the case $c_2 > 0$ and $c_2 < 0$, respectively.

Of course some difficulties come along with these advantages. First, one needs to know K_c. Indeed, with the Aoki phase present, the definition of this quantity is not unique. Second, it is not clear how to extend twisting to odd numbers of flavors, where the $m \leftrightarrow -m$ symmetry is broken by anomalies; the negative mass case then corresponds to the strong CP angle Θ being π, where one expects a spontaneous CP violation surviving in the continuum limit. And finally, the mass needs to be larger than the c_2 artifacts. Indeed, as Figs. 9 and 10 suggest, if it is not, then one is really studying the physics of the Aoki phase, not the correct continuum limit. This also has implications for how

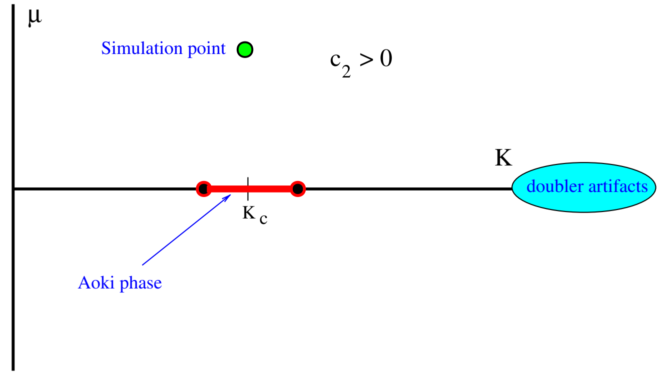

FIG. 9 (color online). Continuing around the Aoki phase with twisted mass. This sketch considers the case $c_2 > 2$ where the parity broken phase extends over a region along the kappa axis.

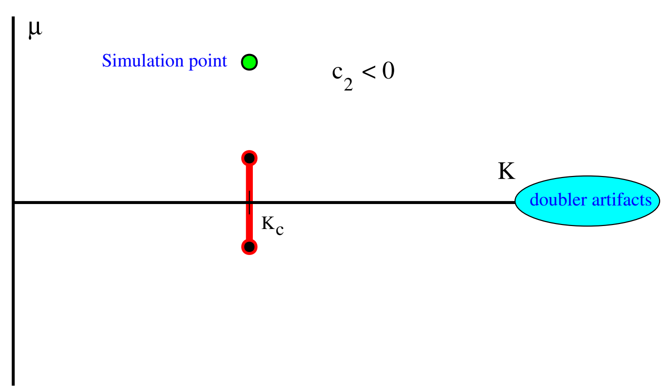

FIG. 10 (color online). As in Fig. 9, but for the case $c_2 < 0$ so the chiral transition on the kappa axis becomes first order.

close to the continuum one must be to study this structure; in particular, one must have β large enough so the Aoki phase does not extend into the doubler region.

VI. DECONFINEMENT AT FINITE TEMPERATURE

Now we bring finite temperature and the deconfinement transition into the discussion. To treat temperature in the path integral formalism, one keeps the temporal extent of the system finite. The temporal boundary conditions should be periodic for bosonic fields and antiperiodic for fermionic ones. The usual order parameter for the transition is the Wilson-Polyakov line. In continuum language this involves an integral that loops around the temporal direction

$$L(\vec{x}) = \mathrm{Re}\,\mathrm{Tr}\mathcal{T}\exp\left(i\int_0^{1/T} A_0(\vec{x})dt\right), \tag{15}$$

where $\mathcal{T}$ denotes time ordering. Of course this quantity needs renormalization for definition. On the lattice this order parameter reduces to the trace of an ordered product of link variables that wraps similarly around the system

$$L(\vec{x}) = \mathrm{Re}\,\mathrm{Tr}\prod_t U_{t+1,t}(\vec{x}) \tag{16}$$

where the product is again time ordered. Physically, one can regard the expectation of L as the exponentiated energy of a stationary source with quark quantum numbers. When $\langle L\rangle$ vanishes, such a source is confined.

The pure gauge theory, where quarks are left out, has a symmetry under taking $L \to e^{2\pi i/3}L$. This can be seen since taking all timelike links at a single fixed time slice and multiplying them by a third root of unity (which is, of course, an $SU(3)$ element) leaves the action invariant. At low temperatures, when confinement is manifest, the vanishing of $\langle L\rangle$ corresponds to the symmetry being unbroken. On the other hand, when the temperature is high, it is expected that the symmetry will be spontaneously broken, corresponding to a deconfined phase. When quarks are introduced with a finite mass, the symmetry ceases to be exact. Physically, dynamical quarks can screen a fixed source, allowing it to have finite energy even if confinement persists. Because of this breaking, the deconfinement transition need not be a true phase transition, and could be only crossover, i.e. a rapid change of behavior over a small but finite range of temperature.

An alternative order parameter for the deconfinement transition is the quark condensate $\langle\bar{\psi}\psi\rangle \sim \langle\sigma\rangle$. At vanishing quark mass, this marker of chiral symmetry breaking is expected to vanish at high temperature, indicative of the evolution of the effective potential $V(\vec{\pi}, \sigma)$ to a state with a single minimum.

Numerous numerical simulations [9] have clearly demonstrated that both $\langle L\rangle$ and $\langle\bar{\psi}\psi\rangle$ do show a rapid change over a single small region of temperature. For large quark mass this transition becomes first order, much as seen in the three-state Potts model, which has the same Z_3 symmetry. At $m = 0$ it is generally expected that the transition becomes second order and is in the same universality class as the $O(4)$ sigma model. However this is not proven and there are some hints [19] that the transition may be first order. Between $m = 0$ and $m = \infty$ it appears that for a large region the transition reduces to a crossover.

Various effective models have been invoked to mimic this behavior [20–23]. These are based on treating the Wilson line as an effective complex field $L(\vec{x})$ in the three spatial dimensions. A simple starting potential is

$$V(L) = \alpha_1|L|^4 + \alpha_2(T_c - T)|L|^2 - \alpha_3\,\mathrm{Re}(L^3) - \alpha_4\,\mathrm{Re}L. \tag{17}$$

Here the $\alpha_1 > 0$ term serves to keep the system stable. The $\alpha_2 > 0$ term is the basic driving term for the transition; when the temperature is below T_c the potential has a unique minimum corresponding to a small or vanishing value for the expectation of the effective field. But when the temperature exceeds T_c, this minimum inverts and we can have a spontaneous symmetry breaking with $\langle L\rangle$ developing a large value. The α_3 term serves to reduce the symmetry to the physical Z_3 rather than the $U(1)$ present with the first two terms alone. This also drives the transition towards first order because of its cubic nature. Finally, the α_4 term is present to represent the fact that dynamical quarks break the symmetry since they can screen other fundamental charges. This term can also soften the transition to a crossover since it allows the order parameter to have a small expectation in the low temperature region. To proceed, rewrite this potential as a function of the lattice parameters, i.e.

$$T_c - T \to \beta_c(K, N_t) - \beta; \tag{18}$$

where $\beta_c(K, N_t)$ is the value of β where deconfinement sets in at a fixed N_t and hopping parameter. Thus we have

$$V(L) = \alpha_1|L|^4 + \alpha_2(\beta_c(K, N_t) - \beta)|L|^2 - \alpha_3\mathrm{Re}(L^3) - \alpha_4\mathrm{Re}L. \tag{19}$$

With effective models for both the chiral fields and the Wilson line, it is natural to unite them into a single effective model involving all fields [24,25]. Combining the previous potentials, consider

$$\begin{aligned} V(\vec{\pi}, \sigma, L) = {}& \lambda(\sigma^2 + \vec{\pi}^2 - v^2)^2 - c_1(1/K - 1/K_c(\beta))\sigma \\ &+ c_2\sigma^2 - \mu\pi_3 + \alpha_1|L|^4 \\ &+ \alpha_2(\beta_c(K, N_t) - \beta)|L|^2 - \alpha_3\,\mathrm{Re}(L^3) \\ &- \alpha_4\,\mathrm{Re}L + \alpha_5|L|^2(\sigma^2 + \vec{\pi}^2). \end{aligned} \tag{20}$$

Here the α_5 term serves to couple the chiral fields with the loop field. Through this term, a jump in $|L|$ can turn off the chiral symmetry breaking. The potential involves two unknown functions: $K_c(\beta)$ and $\beta_c(K, N_t)$. For simplicity in the following discussion, ignore any possible K_c dependence on N_t before deconfinement takes place.

VII. PREDICTIONS

This combined effective potential allows us to sketch qualitatively much of the expected phase diagram for twisted mass Wilson fermions at finite temperature. To understand the structure it is useful to first extract some qualitative features of the continuum physics using the numerically supported fact that increasing the quark mass increases the deconfinement temperature, as sketched schematically in Fig. 11. Since continuum physics is independent of twisting, if we introduce a twisted mass term we obtain an "inverted umbrella" or "conical" structure in m, μ, T space. This structure is symmetric under rotations about the T axis, as sketched in Fig. 12. Of course, as mentioned earlier, the transition need not be a true singularity, but could, for some mass range, just be a rapid crossover.

Now consider an intermediate temperature where the massless theory is deconfined but at some finite mass the theory is still confined. In particular, consider a horizontal slice intersecting the "cone" of Fig. 12. At this temperature the phase diagram in the m, μ plane is simply a circle as sketched in Fig. 13; again, this is an immediate consequence of the continuum physics being independent of twisting. Note that this figure makes it clear that for this two-flavor theory physics at some mass m is equivalent to physics at $-m$, and the two regions are connected by rotations in the m, μ plane.

On the lattice we should expect a similar structure, at least near the continuum limit. However, twisting the mass is no longer an exact symmetry; so, we can expect the circles of equivalent physics to be distorted. Also, in the deconfined region we expect the effective potential to have a single minimum, so that the Aoki phase will wash out after deconfinement. Actually, this disappearance of the Aoki structure at high temperature should be independent of lattice issues such as the sign of c_2. Qualitatively this means that at some fixed β and N_t one should see a structure similar to that sketched in Fig. 14. As mentioned earlier, β will need to be large enough that the Aoki phase structure is well separated from the doubler region. As one comes closer to the continuum limit, this structure should become increasingly precisely an ellipse with constant $\sqrt{\mu^2 + c_1^2(1/K - 1/K_c)^2}$ with c_1 from Eq. (7).

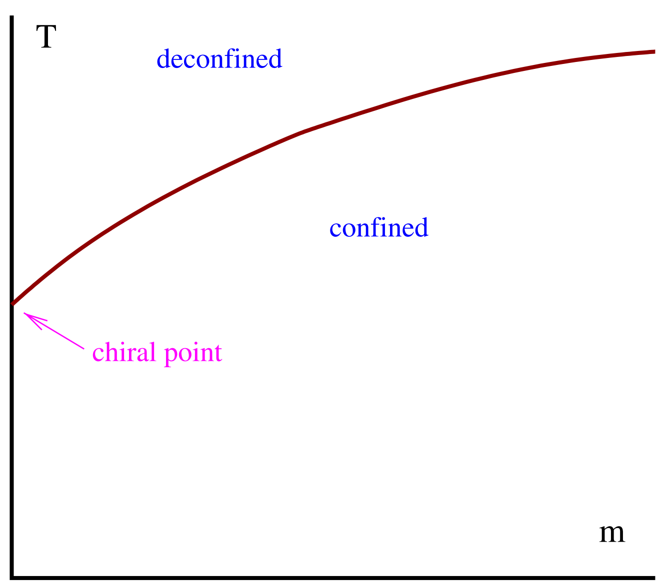

FIG. 11 (color online). The deconfinement transition increases monotonically with the quark mass.

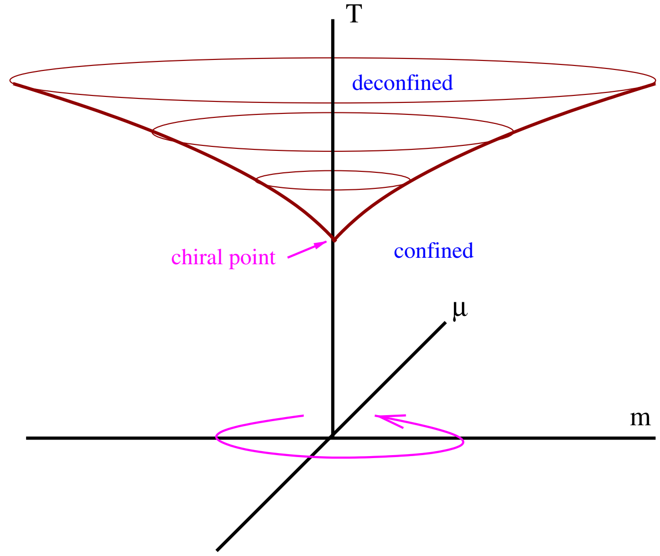

FIG. 12 (color online). Rotating the transition line of Fig. 11 into twisted mass space gives an "inverted umbrella" or conical structure.

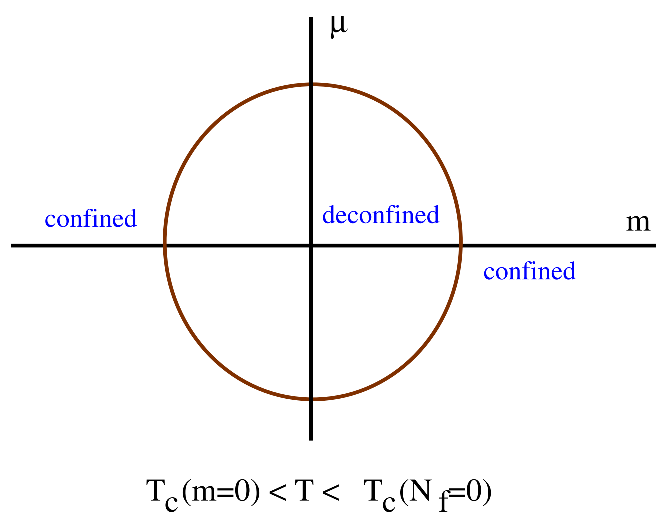

FIG. 13 (color online). In the continuum the deconfinement transition is a simple circle in m, μ space.

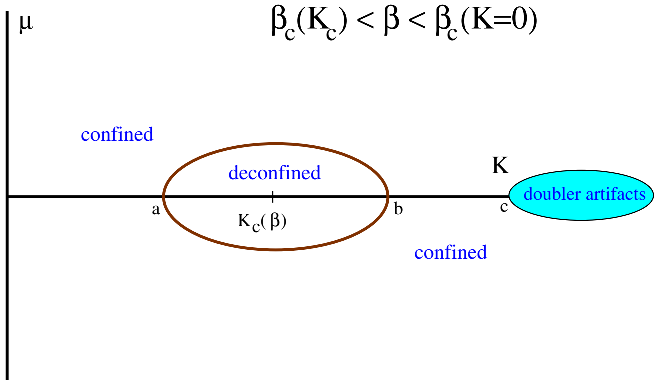

FIG. 14 (color online). At fixed N_t and for a value of β where the infinite mass theory is confined but the massless theory is in the deconfined phase, we expect the deconfined phase to assume an approximately elliptical structure in the $K\mu$ plane.

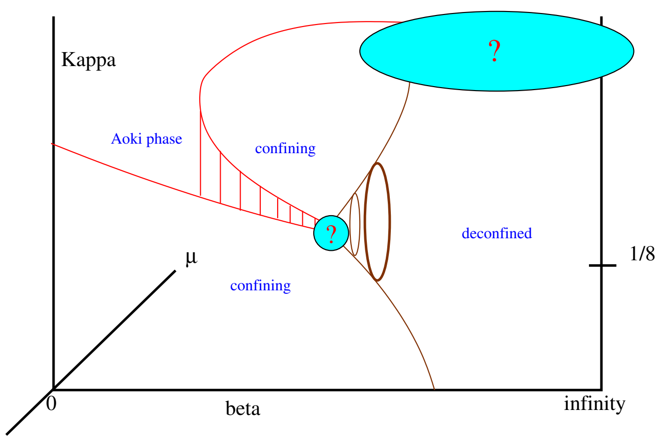

FIG. 15 (color online). The predicted phase diagram in β, K, μ space for fixed N_t. Note how one can smoothly continue from the conventional confining phase to another confining phase in the supercritical hopping region.

Presumably this can be verified in simulations for small μ near the lower deconfinement transition. Working near the upper transition may be more difficult in practice due to small eigenvalues of the Dirac operator for supercritical kappa.

Extending this picture to the full β, K, μ space at fixed N_t, we expect the Aoki phase to dissolve into a conical structure as sketched in Fig. 15. Just how the deconfining cone joins on to the end of the Aoki phase presumably is rather sensitive to dynamical details.

VIII. CONCLUSIONS AND OPEN QUESTIONS

We have seen that Wilson fermions at finite temperature and with a twisted mass term can give rise to a fascinatingly complex phase structure, much of it due to lattice artifacts. This suggests that it could be interesting to study the supercritical hopping region in more detail. In particular a second deconfinement line is expected, as least close to the continuum limit. The twisted mass addition allows us to smoothly connect two deconfined phases. This connection should expose an approximately elliptical structure in the K, μ plane at fixed β.

This picture raises some interesting questions. One concerns the three-flavor theory. In this case the parity broken phase becomes physical. Indeed, three degenerate quarks of negative mass represent QCD with a strong CP angle $\theta = \pi$, for which spontaneous breaking of CP is expected. With three flavors the twisting process is not unique, with possible twists in the λ_3 or λ_8 directions. For example, using only λ_3 would suggest a twisted mass of form $m_u \sim e^{2\pi i/3}$, $m_d \sim e^{-2\pi i/3}$, $m_s \sim 1$.

Another interesting case is one-flavor QCD [16]. In this situation the anomaly removes all chiral symmetry, and the quark condensate loses meaning as an order parameter. The critical value of kappa where the mass gap disappears is decoupled from the point of zero physical quark mass. There is a parity broken phase, but it occurs only at sufficiently negative mass. And from the point of view of twisting the mass, without chiral symmetry there is nothing to twist.

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