

## Aoki Phases in Staggered-Wilson Fermions\*

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We investigate the parity-broken phase (Aoki phase) for staggered-Wilson fermions by using the Gross-Neveu model and the strong-coupling lattice QCD. In the both cases the gap equations indicate the parity-broken phase exists and the pion becomes massless on the phase boundaries. We also show we can take the chiral and continuum limit in the Gross-Neveu model by tuning mass and gauge-coupling parameters. This supports the idea that the staggered-Wilson fermions can be applied to the lattice QCD simulation by taking a chiral limit, as with Wilson fermions.

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## 1. Introduction

Recently staggered-based Wilson fermions were proposed by introducing the taste-splitting mass or the flavored-mass terms into staggered fermions [1, 2, 3, 4]. They can be applied to lattice QCD not only as Wilson fermions but also as an overlap kernel. One possible advantage of these novel fermions called staggered-Wilson and staggered-overlap is reduction of the matrix sizes in the associated Dirac operators, which leads to reduction of numerical costs in lattice QCD simulations. Thus they may be able to overcome the usual naive-fermion-based lattice fermions in lattice QCD [5]. The purpose of this work is reveal properties of staggered Wilson fermions in terms of the parity phase structure (Aoki phase) [6]. The Aoki phase for the staggered-Wilson was first studied in Ref. [4] and the present paper shows further investigation of this topic. The existence of the Aoki phase and the second-order phase boundary in Wilson-type lattice fermions indicates that one can apply them to lattice QCD simulations by tuning a mass parameter to take a chiral limit. Besides, the understanding of the parity-broken phase gives practical information for the application of its overlap and domain-wall versions.

In this paper we elucidate the parity phase structure for staggered-Wilson fermions in the framework of the Gross-Neveu model and the hopping parameter expansion in the strong-coupling lattice QCD. We find the gap equations derived from the both theories show the pion condensate becomes nonzero in some range of the parameters and the pion becomes massless on the phase boundaries. It means the Aoki phase exists and the order of the phase transition is second-order. We also show we can take the chiral continuum limit in the Gross-Neveu model by tuning the mass and the gauge-coupling. These results on the staggered-Wilson fermion indicate we can obtain one- or two-flavor fermions by tuning the mass parameter and perform the lattice QCD simulation with these fermions as in the Wilson fermion. We note the results on the Gross-Neveu model is based on the work by some of the present authors [4, 7] while the results on the strong-coupling lattice QCD are the parts of a work in progress [8].

## 2. Staggered Wilson fermions

We begin with staggered-Wilson fermions in which the flavored-mass terms split the four degenerate tastes in a manner similar to the usual Wilson term. There are two possible types of the flavored-mass terms for staggered fermions as

$$M_f^{(1)} = \varepsilon \sum_{sym} \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4 = (\mathbf{1} \otimes \gamma_5) + O(a), \quad (2.1)$$

$$M_f^{(2)} = \sum_{\mu > \nu} \frac{i}{2\sqrt{3}} \varepsilon_{\mu\nu} \eta_\mu \eta_\nu (C_\mu C_\nu + C_\nu C_\mu) = (\mathbf{1} \otimes \sum_{\mu > \nu} \sigma_{\mu\nu}) + O(a), \quad (2.2)$$

where  $C_\mu = (V_\mu + V_\mu^\dagger)/2$ ,  $(\eta_\mu)_{xy} = (-1)^{x_1 + \dots + x_{\mu-1}} \delta_{x,y}$ ,  $(\varepsilon)_{xy} = (-1)^{x_1 + \dots + x_4} \delta_{x,y}$ ,  $(\varepsilon_{\mu\nu})_{xy} = (-1)^{x_\mu x_\nu} \delta_{x,y}$ , with  $(V_\mu)_{xy} = U_{\mu,x} \delta_{y,x+\mu}$ . In the right hand sides we use the spin-taste representation as  $\mathbf{1} \otimes \gamma_5$ . We refer to  $M_f^{(1)}$  as the Adams-type and  $M_f^{(2)}$  the Hoelbling-type. The former splits the 4 tastes into two with positive ( $m = +1$ ) and the other two with negative ( $m = -1$ ) mass while the latter split them into one with positive ( $m = +2$ ), two with zero ( $m = 0$ ) and the other one with negative mass ( $m = -2$ ).

Now we introduce the Wilson parameter  $r = r\delta_{x,y}$  and shift the mass for the actions as with Wilson fermions. Then the Adams-type staggered-Wilson fermion action is given by

$$S_A = \sum_{xy} \bar{\chi}_x [\eta_\mu D_\mu + r(1 + M_f^{(1)}) + M]_{xy} \chi_y, \quad (2.3)$$

with  $D_\mu = \frac{1}{2}(V_\mu - V_{-\mu})$ . Here  $M$  stands for the usual taste-singlet mass ( $M = M\delta_{x,y}$ ). The Hoelbling-type staggered-Wilson fermion action is given by

$$S_H = \sum_{xy} \bar{\chi}_x [\eta_\mu D_\mu + r(2 + M_f^{(2)}) + M]_{xy} \chi_y. \quad (2.4)$$

In the QCD simulation we will tune the mass parameter  $M$  to take a chiral limit. For some negative values of the mass parameter:  $-1 < M < 0$  for Adams-type and  $-2 < M < 0$  for Hoelbling-type with  $r = 1$ , we obtain two-flavor and one-flavor overlap fermions respectively by using the overlap formula.

The potential problem in lattice QCD with these fermions is the breaking of some discrete symmetries as the shift symmetry caused by the flavored-mass terms [2, 3]. There has not yet been a consensus on whether it does harm to lattice QCD with staggered-Wilson fermions. We can answer this question partly by studying the Aoki phase since a clear symptom is expected to appear in the phase structure if the symmetry breaking ruins the essential properties of QCD. In the following sections we will find the Aoki phase structure in the staggered-Wilson fermion is qualitatively similar to the original Wilson one and there is no disease.

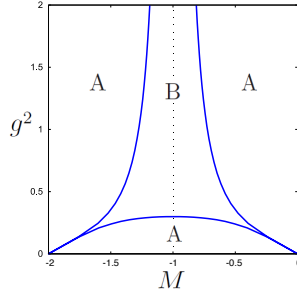
### 3. Gross-Neveu model

We first investigate the parity phase diagram for staggered-Wilson fermions by using the  $d = 2$  Gross-Neveu model as a toy model of QCD. To study the pion condensate we generalize the usual staggered Gross-Neveu model to the one with the  $\gamma_5$ -type 4-point interaction, which is given by

$$S = \frac{1}{2} \sum_{n,\mu} \eta_\mu \bar{\chi}_n (\chi_{n+\mu} - \chi_{n-\mu}) + \sum_n \bar{\chi}_n (M + r(1 + M_f)) \chi_n - \frac{g^2}{2N} \sum_{\mathcal{N}} \left[ \left( \sum_A \bar{\chi}_{2\mathcal{N}+A} \chi_{2\mathcal{N}+A} \right)^2 + \left( \sum_A i(-1)^{A_1+A_2} \bar{\chi}_{2\mathcal{N}+A} \chi_{2\mathcal{N}+A} \right)^2 \right], \quad (3.1)$$

where the two-dimensional coordinate is defined as  $n = 2\mathcal{N} + A$  with sublattices  $A = (A_1, A_2)$  ( $A_{1,2} = 0, 1$ ). In this model  $\chi_n$  is a  $N$ -component one-spinor  $(\chi_n)_j$  ( $j = 1, 2, \dots, N$ ) where  $\bar{\chi}\chi = \sum_{j=1}^N \bar{\chi}_j \chi_j$ .  $(-1)^{A_1+A_2}$  corresponds to  $\Gamma_{55} = \gamma_5 \otimes \gamma_5$  in the spinor-taste expression while  $\eta_\mu = (-1)^{n_1 + \dots + n_{\mu-1}}$  corresponds to  $\gamma_\mu$ . In this dimension the Adams-type and Hoelbling-type flavored-mass terms coincide and there is only one type  $M_f = \Gamma_5 \Gamma_{55} \sim \mathbf{1} \otimes \gamma_5 + O(a)$  with  $\Gamma_5 = -i\eta_1 \eta_2 \sum_{\text{sym}} C_1 C_2$ . This mass term assigns the positive mass ( $m = +1$ ) to one taste and the negative mass ( $m = -1$ ) to the other. With bosonic auxiliary fields  $\sigma_{\mathcal{N}}, \pi_{\mathcal{N}}$  leading to  $\sigma$ -meson and  $\pi$ -meson fields, the action is rewritten as

$$S = \frac{1}{2} \sum_{n,\mu} \eta_\mu \bar{\chi}_n (\chi_{n+\mu} - \chi_{n-\mu}) + \sum_n \bar{\chi}_n M_f \chi_n + \frac{N}{2g^2} \sum_{\mathcal{N}} \left( (\sigma_{\mathcal{N}} - 1 - M)^2 + \pi_{\mathcal{N}}^2 \right) + \sum_{\mathcal{N},A} \bar{\chi}_{2\mathcal{N}+A} (\sigma_{\mathcal{N}} + i(-1)^{A_1+A_2} \pi_{\mathcal{N}}) \chi_{2\mathcal{N}+A}, \quad (3.2)$$



**Figure 1:** Aoki phase structure for the staggered-Wilson fermion in the Gross-Neveu model. A stands for a parity symmetric phase and B for Aoki phase.

where we take  $r = 1$  as the Wilson parameter. After integrating the fermion field, the partition function and the effective action with these auxiliary fields(meson fields) are given by

$$Z = \int \mathcal{D}\sigma_{\mathcal{N}} \mathcal{D}\pi_{\mathcal{N}} e^{-N S_{\text{eff}}(\sigma, \pi)}, \quad S_{\text{eff}} = \frac{1}{2g^2} \sum_{\mathcal{N}} ((\sigma_{\mathcal{N}} - 1 - M)^2 + \pi_{\mathcal{N}}^2) - \text{Tr} \log D, \quad (3.3)$$

with  $D_{n,m} = (\sigma_{\mathcal{N}} + i(-1)^{A_1+A_2} \pi_{\mathcal{N}}) \delta_{n,m} + \frac{\eta_{\mu}}{2} (\delta_{n+\mu,m} - \delta_{n-\mu,m}) + (M_f)_{n,m}$ . In the large N limit the partition function is given by the saddle point of the action as  $Z = e^{-S_{\text{eff}}(\sigma_0, \pi_0)}$  with the translation-invariant solutions  $\sigma_0, \pi_0$  satisfying the saddle-point equations  $\frac{\delta S_{\text{eff}}(\sigma_0, \pi_0)}{\delta \sigma_0} = \frac{\delta S_{\text{eff}}(\sigma_0, \pi_0)}{\delta \pi_0} = 0$ . After some calculation process to derive the fermion determinant [4] we obtain the concrete forms of the saddle-point equations in the momentum space

$$\frac{\sigma_0 - 1 - M}{g^2} = 4 \int \frac{dk^2}{(2\pi)^2} \frac{\sigma_0(\sigma_0^2 + \pi_0^2 + s^2) - c_1^2 c_2^2 \sigma_0}{((\sigma_0 + c_1 c_2)^2 + \pi_0^2 + s^2)((\sigma_0 - c_1 c_2)^2 + \pi_0^2 + s^2)}, \quad (3.4)$$

$$\frac{\pi_0}{g^2} = 4 \int \frac{dk^2}{(2\pi)^2} \frac{\pi_0(\sigma_0^2 + \pi_0^2 + s^2) + c_1^2 c_2^2 \pi_0}{((\sigma_0 + c_1 c_2)^2 + \pi_0^2 + s^2)((\sigma_0 - c_1 c_2)^2 + \pi_0^2 + s^2)}, \quad (3.5)$$

with  $c_{\mu} = \cos k_{\mu}/2$  and  $s_{\mu} = \sin k_{\mu}/2$ . Now what we are interested in is the parity phase diagram in this theory. The parity phase boundary  $M_c(g^2)$  is derived by imposing  $\pi_0 = 0$  in (3.4)(3.5) after the overall  $\pi_0$  being removed in the second one. Then the gap equations are given by

$$\frac{1 + M_c}{g^2} = 4 \int \frac{dk^2}{(2\pi)^2} \frac{2c_1^2 c_2^2 \sigma_0}{((\sigma_0 + c_1 c_2)^2 + \pi_0^2 + s^2)((\sigma_0 - c_1 c_2)^2 + \pi_0^2 + s^2)}, \quad (3.6)$$

$$\frac{1}{g^2} = 4 \int \frac{dk^2}{(2\pi)^2} \frac{\sigma_0^2 + s^2 + c_1^2 c_2^2}{((\sigma_0 + c_1 c_2)^2 + \pi_0^2 + s^2)((\sigma_0 - c_1 c_2)^2 + \pi_0^2 + s^2)}. \quad (3.7)$$

By removing  $\sigma_0$  in these equations, we derive the phase boundary  $M_c(g^2)$ . The result is shown in Fig. 1. It indicates the parity phase structure in the staggered-Wilson fermion is qualitatively similar to the usual Wilson case [6] reflecting the mass splitting of tastes given by the flavored mass. We also check the pion mass becomes zero on the second order phase boundary as

$$m_{\pi}^2 \propto \left\langle \frac{\delta^2 S_{\text{eff}}}{\delta \pi_{\mathcal{N}} \delta \pi_{\mathcal{M}}} \right\rangle \Big|_{M=M_c} = V \frac{\delta^2 \tilde{S}_{\text{eff}}}{\delta^2 \pi_0^2} \Big|_{M=M_c} = 0. \quad (3.8)$$

where  $S_{\text{eff}} = V \tilde{S}_{\text{eff}}$  with  $V$  being the volume.

We next consider the chiral and continuum limit of the staggered-Wilson Gross-Neveu models. The strategy is to expand the fermion determinant in the effective potential in Eq. (3.3) with respect to the lattice spacing  $a$ . After some calculations (See details in Ref. [4]) we obtain the effective potential remaining in the limit  $a \rightarrow 0$ ,

$$\begin{aligned} \tilde{S}_{\text{eff}} = & - \left( \frac{M+1/a}{g_\sigma^2} + \frac{2}{a}C_1 \right) \sigma_0 + \left( \frac{1}{2g_\pi^2} - \tilde{C}_0 + \frac{1}{\pi} \log 4a^2 \right) \pi_0^2 \\ & + \left( \frac{1}{2g_\sigma^2} - \tilde{C}_0 + 2C_2 + \frac{1}{\pi} \log 4a^2 \right) \sigma_0^2 + \frac{1}{\pi} (\sigma_0^2 + \pi_0^2) \log \frac{\sigma_0^2 + \pi_0^2}{e}. \end{aligned} \quad (3.9)$$

with the three numbers as  $\tilde{C}_0 = 1.177$ ,  $C_1 = -0.896$  and  $C_2 = 0.404$ . Here the chiral symmetry is encoded as the rotational symmetry between  $\sigma_0$  and  $\pi_0$ ; thus, the chiral limit means restoring this symmetry by tuning the parameters. In this model we need introduce two independent coupling constants  $g_\sigma^2$  and  $g_\pi^2$  to restore the symmetry although the necessity of two couplings is just a model artifact. The tuned point for the chiral limit without  $O(a)$  corrections is

$$M = -\frac{2g_\sigma^2}{a}C_1 - 1, \quad g_\pi^2 = \frac{g_\sigma^2}{4C_2g_\sigma^2 + 1}, \quad (3.10)$$

To take the continuum limit we introduce the  $\Lambda$ -parameter as  $2a\Lambda = \exp \left[ \frac{\pi}{2}\tilde{C}_0 - \pi C_2 - \frac{\pi}{4g_\sigma^2} \right]$ . Then the coupling renormalization for the chiral and continuum limit is given by

$$\frac{1}{2g_\sigma^2} = \tilde{C}_0 - 2C_2 + \frac{1}{\pi} \log \left( \frac{1}{4\Lambda^2 a^2} \right), \quad \frac{1}{2g_\pi^2} = \tilde{C}_0 + \frac{1}{\pi} \log \left( \frac{1}{4\Lambda^2 a^2} \right). \quad (3.11)$$

where we keep  $\Lambda$  finite when taking the continuum limit  $a \rightarrow 0$ . Finally the renormalized effective potential in the chiral and continuum limit is given by

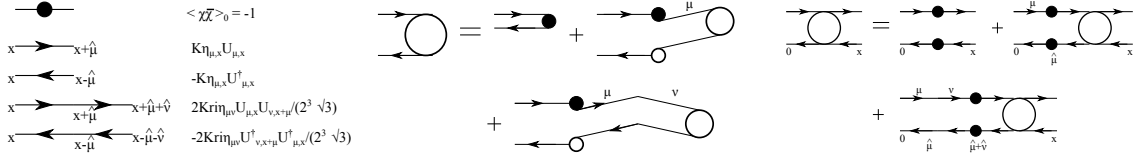
$$\tilde{S}_{\text{eff}} = \frac{1}{\pi} (\sigma_0^2 + \pi_0^2) \log \frac{\sigma_0^2 + \pi_0^2}{e\Lambda^2}, \quad (3.12)$$

This wine-bottle potential yields the spontaneous breaking of the rotational symmetry. We have shown that the chirally-symmetric continuum limit can be taken by fine-tuning a mass parameter and two coupling constants in the staggered-Wilson Gross-Neveu model. Considering that the necessity of the two coupling constants is just a model artifact, this result indicates we can take a chiral limit by tuning only the mass parameter as in the Wilson fermion. Indeed, our results on the chiral and continuum limit for staggered-Wilson are almost the same as the Wilson case [6].

#### 4. Strong-coupling QCD

In this section we investigate the Aoki phase structure in lattice QCD with the staggered-Wilson fermion in the framework of the hopping parameter expansion (HPE) in the strong-coupling regime. For simplicity we concentrate on the Hoelbling-type lattice fermion here, but we can also make the same analysis in a parallel way for the Adams-type fermion. To perform the HPE for the Hoelbling-type fermion, we rewrite the action (2.4) by redefining  $\chi \rightarrow \sqrt{2K}\chi$  with  $K = 1/[2(M+2r)]$ ,

$$S = \sum_x \tilde{\chi}_x \chi_x + 2K \sum_{x,y} \tilde{\chi}_x (\eta_\mu D_\mu)_{xy} \chi_y + 2Kr \sum_{x,y} \tilde{\chi}_x (M_f)_{xy} \chi_y. \quad (4.1)$$



**Figure 2:** (Left):Feynman rules for the HPE. (Center):one point function. (Right):two point function.

In Fig. 2(Left) we write down the Feynman rules in the HPE for this fermion. Now we derive chiral and pion condensates from the one-point function of the meson operator ( $\mathcal{M}_x = \bar{\chi}_x \chi_x$ ) in the mean-field approximation. The equation for the one-point function is obtained in a self-consistent way as shown in Fig. 2(Center),

$$-\Sigma_x \equiv \langle \mathcal{M}_x \rangle = \langle \mathcal{M}_x \rangle_0 + 2K^2 \sum_{\mu} \Sigma_{x+\hat{\mu}} \Sigma_x - 2 \cdot \frac{1}{24} (Kr)^2 \sum_{\mu \neq \nu} \Sigma_x \Sigma_{x+\hat{\mu}+\hat{\nu}}, \quad (4.2)$$

where we drop the link variable since we work in the strong-coupling limit. Other diagrams are found to vanish due to the grassman properties and the cancellation between the diagrams. Here we solve this as a self-consistent equation for the condensate  $\Sigma$  within the mean-field approximation. For our purpose we assume  $\Sigma_x = \sigma_x + i\varepsilon_x \pi_x$ , where  $\sigma_x$  and  $\pi_x$  correspond to the chiral and pion condensates. We substitute this form of  $\Sigma_x$  in Eq. (4.2) and obtain the self-consistent equation

$$-(\sigma + i\varepsilon_x \pi) = -1 + 2K^2 \cdot 4 (\sigma^2 + \pi^2) - 2 \cdot \frac{1}{24} (Kr)^2 \cdot 4 \cdot 3 (\sigma + i\varepsilon_x \pi)^2, \quad (4.3)$$

which yields  $-\sigma = -1 + 16K^2 \pi^2$  and  $-i\pi = -8K^2 \cdot 2i\sigma \pi$ . Here we have set  $r = 2\sqrt{2}$  for simplicity. We have two solutions depending on whether  $\pi = 0$  or  $\pi \neq 0$ : For  $\pi = 0$  we have a trivial solution  $\sigma = 1$ . For  $\pi \neq 0$  we have a non-trivial solution as

$$\sigma = \frac{1}{16K^2}, \quad \pi = \pm \sqrt{\frac{1}{16K^2} \left(1 - \frac{1}{16K^2}\right)}. \quad (4.4)$$

In this solution the pion condensate is non-zero and the  $\pm$  signs indicate the spontaneous parity breaking. This parity-broken phase (Aoki phase) appears in the range of the hopping parameter or the mass parameter as  $|K| > 1/4$  or equivalently  $-4\sqrt{2} - 2 < M < -4\sqrt{2} + 2$ .

We next discuss the two-point function of the meson operator  $\mathcal{S}(0, x) \equiv \mathcal{M}_0 \mathcal{M}_x$ . From Fig. 2(Right) we derive the following equation for two point function. (other diagrams vanish again.)

$$\begin{aligned} \mathcal{S}(0, x) \equiv \langle \bar{\chi}_0^a \chi_0^a \bar{\chi}_x^b \chi_x^b \rangle &= -\delta_{0x} N_c + K^2 \sum_{\pm\mu} \langle \bar{\chi}_{\hat{\mu}}^a \chi_{\hat{\mu}}^a \bar{\chi}_x^b \chi_x^b \rangle \\ &+ \left(2Kr \frac{1}{2^3\sqrt{3}}\right)^2 \sum_{\substack{\pm\mu, \pm\nu \\ (\mu \neq \nu)}} \langle \bar{\chi}_{\hat{\mu}+\hat{\nu}}^a \chi_{\hat{\mu}+\hat{\nu}}^a \bar{\chi}_x^b \chi_x^b \rangle. \end{aligned} \quad (4.5)$$

Then the self-consistent equation for  $\mathcal{S}$  is given in the momentum space as

$$\begin{aligned} \mathcal{S}(p) &= -N_c + \left[ -K^2 \sum_{\mu} (e^{-ip_{\mu}} + e^{ip_{\mu}}) \right. \\ &\left. + \left(2Kr \frac{1}{2^3\sqrt{3}}\right)^2 \sum_{\mu \neq \nu} (e^{-i(p_{\mu}+p_{\nu})} + e^{i(p_{\mu}+p_{\nu})} + e^{-i(p_{\mu}-p_{\nu})} + e^{i(p_{\mu}-p_{\nu})}) \right] \mathcal{S}(p). \end{aligned} \quad (4.6)$$

We finally obtain the meson propagator as

$$\mathcal{S}(p) = N_c \left[ -2K^2 \sum_{\mu} \cos p_{\mu} + 4 \left( 2Kr \frac{1}{2^3 \sqrt{3}} \right)^2 \sum_{\mu \neq \nu} \cos p_{\mu} \cos p_{\nu} - 1 \right]^{-1}. \quad (4.7)$$

The pole of  $\mathcal{S}(p)$  gives the meson mass. Remembering  $\gamma_5$  in the staggered fermion is given by  $\varepsilon_x = (-1)^{x_1 + \dots + x_4}$  and the pion operator is given by  $\pi_x = \bar{\chi}_x i \varepsilon_x \chi_x$ , it is obvious that the momentum of the pion should be measured from the shifted origin  $p = (\pi, \pi, \pi, \pi)$ . Thus we set  $p = (im_{\pi}a + \pi, \pi, \pi, \pi)$  for  $1/\mathcal{S}(p) = 0$  in (4.7), which gives the pion mass  $m_{\pi}$  as

$$\cosh(m_{\pi}a) = 1 + \frac{1 - 16K^2}{6K^2}. \quad (4.8)$$

In this result the pion mass becomes tachyonic in the range  $|K| > 1/4$ . It indicates there occurs a phase transition between parity-symmetric and broken phases at  $|K| = 1/4$ , which is consistent with the result on the condensates in Eq. (4.4). The reason why the pion has tachyonic mass in the parity-broken phase in the HPE is the vacuum we work on is not proper. To derive the pion mass in this phase we need to analyze the effective potential, which we will show in the full paper [8].

## 5. Summary

In this paper we study the Gross-Neveu model and the strong-coupling lattice QCD with staggered Wilson fermions with emphasis on the Aoki phase structure. We have shown the parity broken phase and the second order phase boundary exist in the staggered-Wilson fermions as with the Wilson fermion. Our results indicate that we can apply the staggered Wilson fermions to lattice QCD simulations by mass parameter tuning, where we derive one-flavor fermion in the Hoelbling-type and two-flavor fermions in the Adams-type fermions. These results also indirectly suggest the applicability of the staggered overlap and staggered domain-wall fermions to lattice QCD. We note our results on the Aoki phase diagram exhibit no diseases due to a discrete symmetry breaking. It implies the symmetry breaking, for example breaking of the shift symmetry, does no harm to a QCD simulation with the staggered-Wilson fermion, which is consistent with the results in the lattice perturbation in [2, 3]. In the full paper we also discuss the parity-flavor phase structure in the case of the two flavor QCD. The analysis is parallel, and we will find a spontaneous parity-flavor breaking as in the Wilson fermion.

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