The lattice and quantized Yang-Mills theory

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Quantized Yang-Mills fields lie at the heart of our understanding of the strong nuclear force. To understand the theory at low energies, we must work in the strong coupling regime. The primary technique for this is the lattice. While basically an ultraviolet regulator, the lattice avoids the use of a perturbative expansion. I discuss the historical circumstances that drove us to this approach, which has had immense success, convincingly demonstrating quark confinement and obtaining crucial properties of the strong interactions from first principles.

Keywords: Lattice Gauge Theory; lattice; quarks; confinement; Yang-Mills

1. Introduction

As we have been hearing throughout this meeting, the Yang-Mills theory[1] was developed in an attempt to generalize the gauge symmetry of electromagnetism to the non-Abelian SU(2) symmetry of isospin. Remarkably, this simple idea has developed into a core ingredient of all modern theories of elementary particles. With the particular application to the strong interactions, quarks interact by exchanging non-Abelian gauge gluons. This gives rise to some rather unique issues. In particular, asymptotic freedom and dimensional transmutation imply that low energy physics is controlled by large effective coupling constants. Long distance phenomena, such as chiral symmetry breaking and quark confinement, lie outside the realm of accessibility to the traditional Feynman diagram approach. This has driven theorists to new approaches, amongst which the lattice has proven the most successful.

In this talk I will present an overview of what motivated the lattice approach and how it grew to become the dominant technique to study non-perturbative effects in quantum field theory. Much of this presentation is adapted from a previous review[2]. Along the way we will see that there are both practical and fundamental issues with the lattice method. On the practical side, quantitative computer calculations are now routine for non-perturbative effects in the strong interactions. On the more conceptual side, the lattice gives deep insights into the workings of relativistic field theory.

2. Before the lattice

I begin with the situation in particle physics in the late 60's, when I was a graduate student. Quantum-electrodynamics was the model field theory, with immense success. While hard calculations remained, and indeed still remain, the feeling was that this theory was understood. Some subtle conceptual issues do still remain, such as the likely breakdown of the perturbative expansion at ultra high energies.

These were the years when the "eight-fold way" for describing multiplets of particles had gained widespread acceptance[3]. The idea of "quarks" was around, but with considerable caution about assigning them any physical reality; were they nothing but a useful mathematical construct? A few insightful theorists were working on the weak interactions, and the basic electroweak unification was emerging[4,5]. The SLAC experiments were observing substantial inelastic electron-proton scattering at large angles. This was quickly interpreted as evidence for substructure, and the idea of "partons" became popular. While there were speculations on connections between quarks and partons, people tended to be rather cautious about pushing this too hard.

A crucial feature of the time was the failure of extension of quantum electrodynamics to a meson-nucleon field theory. The pion-nucleon analog of the electromagnetic coupling had a value about 15, in comparison with the 1/137 of QED. This meant that higher order corrections to perturbative processes were substantially larger than the initial calculations. There was no known small parameter in which to expand.

In frustration over this situation, much of the particle theory community abandoned traditional quantum field theoretical methods and explored the possibility that particle interactions might be completely determined by fundamental postulates such as analyticity and unitarity. This "S-matrix" approach raised the deep question of just "what is elementary?" A delta baryon might be regarded as a combination of a proton and a pion, but it would be just as correct to regard the proton as a bound state of a pion with a delta. All particles were to be thought of as bound together by exchanging themselves[6]. These "dual" views of the basic objects of the theory have evolved into many of the ideas of string theory.

3. The birth of QCD

In the early 1970's, partons were increasingly identified with quarks. This shift was pushed by two dramatic theoretical accomplishments. First was the proof of renormalizability for Yang-Mill's theories[7], giving confidence that these elegant mathematical structures[1] might indeed have something to do with reality. Second was the discovery of asymptotic freedom, the fact that interactions in Yang-Mills theories become weaker at short distances[8,9]. Indeed, this was quickly connected with the point-like structures hinted at in the SLAC experiments. Out of these ideas evolved QCD, the theory of quark confining dynamics.

The viability of this picture depends upon the concept of "confinement." While

there exists strong evidence for quark substructure, no free quarks have ever been observed. This is particularly puzzling given the nearly free nature of their interactions inside the nucleon. Indeed, the question of "what is elementary?"reappears. Are the fundamental objects the physical particles we see in the laboratory or are they these postulated quarks and gluons?

Struggling with this paradox led to the now standard flux-tube picture of confinement. The eight gluons are analogues of photons except that they carry "charge" with respect to each other. Gluons would presumably be massless like the photon were it not for confinement. But a massless charged particle would be a rather peculiar object. Indeed, what happens to the self energy in the electric fields around a gluon? Such questions naturally lead to a conjectured instability of the æther that removes zero mass gluons from the physical spectrum. This should be done in a way that does not violate Gauss's law. Note that a Coulombic $1/r^2$ field is a solution of the equations of a massless field, not a massive one. Without massless particles in the spectrum, such a spreading of the gluonic flux is not allowed since it cannot satisfy the appropriate equations in the weak field limit. According to Gauss's law, the field lines emanating from a quark cannot simply end. Instead of spreading in an inverse square manner, the gluo-electric flux lines cluster together, forming a tube emanating from the quark and ultimately ending on an anti-quark as sketched in Fig. 1. This structure should be regarded as a real physical object, which grows in length as the quark and anti-quark are pulled apart. The resulting force is constant at long distance, and is measured via the spectrum of high angular-momentum states, organized into the famous "Regge trajectories." In physical units, the flux tube pulls with a tension of about 14 tons.

In essence, the reason a quark cannot be isolated is similar to the fact that a piece of string cannot have just one end. Of course a piece of string can't have three ends either. This is resolved by the underlying $SU(3)$ group theory, wherein three fundamental charges can form a neutral object. It is important to emphasize that the confinement phenomenon cannot be seen in perturbation theory; when the coupling is turned off, the spectrum becomes free quarks and gluons, dramatically different than the pions and protons of the interacting theory.

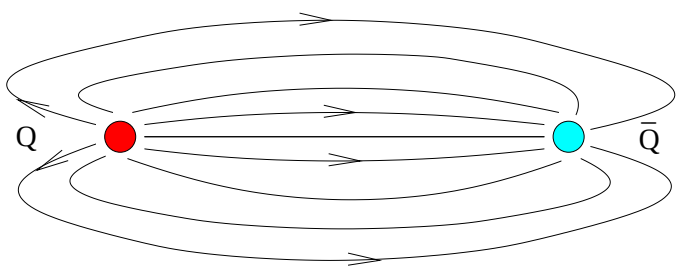

Fig. 1. A tube of gluonic flux connects quarks and anti-quarks. The strength of this string is 14 tons.

4. The 70's revolution

The discoveries related to the Yang-Mills theory were just the beginning of a revolutionary period in particle physics. Perhaps the most dramatic event was the discovery of the J/ψ particle[10]. The interpretation of this object and its partners as bound states of heavy quarks provided what might be regarded as the hydrogen atom of QCD. The idea of quarks became inescapable; field theory was reborn. The $SU(3)$ non-Abelian gauge theory of the strong interactions was combined with the recently developed electroweak theory to become the durable "standard model."

This same period also witnessed several remarkable events on a more theoretical front. Non-linear effects in classical field theories were shown to have deep consequences for their quantum counterparts. Classical "lumps" represented a new way to get particles in a quantum field theory[11]. Much of the progress here was in two dimensions, where techniques such as "bosonization" showed equivalences between theories of drastically different appearance. A boson in one approach might appear as a bound state of fermions in another, but in terms of the respective Lagrangian approaches, they were equally fundamental. Again, we were faced with the question of "what is elementary?"

The interest in classical solutions quickly led to the discovery[12] of "pseudoparticles" or "instantons," solutions of the four dimensional Yang-Mills theory in Euclidean space time. These turned out to be intimately related to the famous anomalies in current algebra, and gave a simple mechanism to generate the masses of such particles as the η'[13]. These effects are inherently non-perturbative, having an explicit exponential dependence in the inverse coupling.

This slew of discoveries had deep implications: field theory had many aspects that could not be seen via the traditional analysis of Feynman diagrams. This has crucial consequences for practical calculations. Field theory is notorious for divergences requiring regularization. The bare mass and charge are divergent quantities. They are not physical observables. For practical calculations, a "regulator" is required to tame the divergences, and when physical quantities are related to each other, any regulator dependence should drop out.

The need for controlling infinities had been known since the early days of QED. But all regulators in common use were based on Feynman diagrams; one would calculate until a divergent diagram appeared, and that diagram was then cut off. Numerous schemes were devised for this purpose, ranging from the Pauli-Villars approach[14] to the forest formulae[15] to dimensional regularization[16]. But with the increasing realization that non-perturbative phenomena were crucial, it was becoming clear that we needed a "non-perturbative" regulator, independent of Feynman diagrams.

5. The lattice

The necessary tool appeared with Wilson's lattice theory. He originally presented this as an example of a model exhibiting confinement. The strong coupling ex-

pansion has a non-zero radius of convergence, allowing a rigorous demonstration of confinement, albeit in an unphysical limit. The resulting spectrum has exactly the desired properties; only gauge singlet bound states of quarks and gluons can propagate.

This was not the first time that the basic structure of lattice gauge theory had been written down. A few years earlier, Wegner[17] presented a Z_2 lattice gauge model as an example of a system possessing a phase transition but not exhibiting any local order parameter. In his thesis, Jan Smit[18] described using a lattice to formulate gauge theories outside of perturbation theory. Very quickly after Wilson's suggestion, Balian, Drouffe, and Itzykson[19–21] explored an amazingly wide variety of aspects of these models.

To reiterate, the primary role of the lattice is to provide a non-perturbative regulator. Space-time is not really meant to be a crystal; the lattice is a mathematical trick. It provides a minimum wavelength through the lattice spacing a, *i.e.* a maximum momentum of π/a. Path summations become well defined ordinary integrals. By avoiding the convergence difficulties of perturbation theory, the lattice provides a route towards a rigorous definition of a quantum field theory as a limiting process.

This approach had a marvelous side effect. After discreetly making the system discrete, the lattice system becomes sufficiently well defined to be placed on a computer[22]. This was fairly straightforward, and came at the same time that computers were growing rapidly in power. Indeed, numerical simulations and computer capabilities have continued to grow together, making these efforts the mainstay of modern lattice gauge theory.

6. Gauge fields and phases

As formulated by Wilson, the lattice cutoff is remarkable in remaining true to many of the underlying concepts of a gauge theory. At the most simplistic level, a Yang-Mills theory is simply electrodynamics embellished with isospin symmetry. By working directly with elements of the gauge group, this is inherent in lattice gauge theory from the start.

At another level, a gauge theory is a theory of phases acquired by a particle as it passes through space time. Using group elements on links directly gives this connection, with the phase associated with some world-line being the product of these elements along the path in question. Ror the Yang-Mills theory, the concept of "phase" becomes a rotation in the internal symmetry group.

At a still deeper level, a gauge theory is a theory with a local symmetry. With the Wilson action being formulated in terms of products of group elements around closed loops, this symmetry remains exact even with the regulator in place.

In perturbative discussions, the local symmetry forces one to adopt a gauge-fixing prescription to remove a formal infinity on integrating over gauges. The lattice formulation, in contrast, uses a compact representation for the group elements, making the integration over all gauges finite. For gauge invariant observables, no

gauge fixing is required. While gauge fixing can still be done, and must be introduced to study such conventional gauge-variant quantities such as gluon or quark propagators.

One property of continuum gauge theory that the lattice approach violates involves transformations under Lorentz transformations. In a continuum theory the basic vector potential can change under a gauge transformation when transforming between frames[23–25]. The lattice, of course, breaks Lorentz invariance, and thus this concept loses meaning until the continuum limit is taken.

7. The Wilson action

The concept of gauge fields representing path dependent phases leads directly to the conventional lattice formulation. We approximate a general quark world-line by a set of hoppings lying along lattice bonds, as sketched in Fig. 2. We then introduce the gauge field as group valued matrices on these bonds. Thus the gauge fields form a set of $SU(3)$ matrices, one such associated with every nearest neighbor bond on our four-dimensional hyper-cubic lattice.

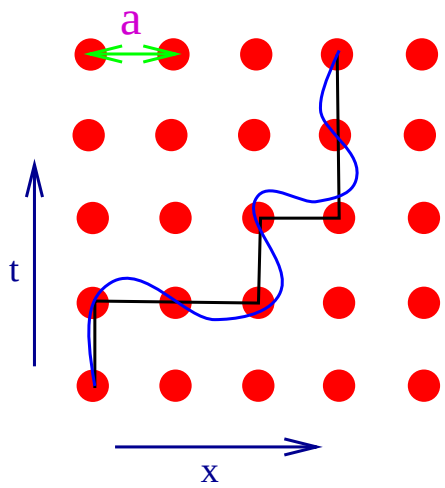

Fig. 2. In lattice gauge theory the world-line describing the motion of a quark through space-time is approximated by a sequence of discrete hops. On each of these hops the quark wave function picks up a "phase" described by the gauge fields. For the strong interactions, this phase is a unitary matrix in the group $SU(3)$.

In terms of these matrices, the gauge field dynamics takes a simple natural form. In analogy with regarding electromagnetic flux as the generalized curl of the vector potential, we are led to identify the flux through an elementary square, or "plaquette," on the lattice with the phase factor obtained on running around that plaquette; see Fig. 3. Spatial plaquettes represent the "magnetic" effects and plaquettes with one time-like direction give the "electric" fields. This motivates the

conventional "action" used for the gauge fields as a sum over all the elementary squares of the lattice. Around each square we multiply the phases and to get a real number we take the real part of the trace

$$S_g = \sum_p \text{Re Tr} \prod_{l \in p} U_l \sim \int d^4x \; E^2 + B^2. \tag{1}$$

Here the fundamental squares are denoted p and the links l. As we are dealing with non-commuting matrices, the product around the square is meant to be ordered, while because of the trace, the starting point of this ordering drops out.

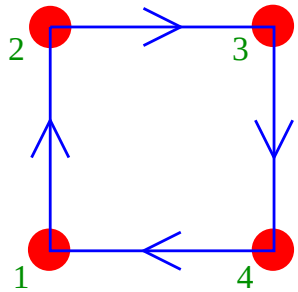

Fig. 3. Analogous to Stoke's law, the flux through an elementary square of the lattice is found from the product of gauge matrices around that square. The dynamics is determined by adding the real part of the trace of this product over all elementary squares. This "action" is inserted into a "path integral." The resulting construction is formally equivalent to a partition function for a system of "spins" existing in the group $SU(3)$.

To formulate the quantum theory of this system one usually uses the Feynman path integral. To construct this, exponentiate the action and integrate over all dynamical variables

$$Z = \int (dU) e^{-\beta S}, \tag{2}$$

where the parameter β controls the bare coupling. This converts the three space dimensional quantum field theory of gluons into a classical statistical mechanical system in four space-time dimensions. Such a many-degree-of-freedom statistical system cries out for Monte Carlo simulation, which now dominates the field of lattice QCD. Note the close analogy with a magnetic system; we could think of our matrices as "spins" interacting through a four spin coupling expressed in terms of the plaquettes.

The formulation is conventionally taken in Euclidean four-dimensional space. In effect this replaces the time evolution operator e^{-iHt} by e^{-Ht}. Despite involving the same Hamiltonian H, excited states are inherently suppressed and information on high energy scattering is particularly hard to extract. However low energy states and matrix elements are the natural physical quantities to explore numerically. This is the bread and butter of the lattice theorist. Indeed, the simulations reproduce

the qualitative spectrum of stable hadrons quite well[26]. Matrix elements currently under intense study are playing a crucial role in ongoing tests of the standard model of particle physics.

8. A paucity of parameters

It is important to emphasize one of the most remarkable aspects of QCD: the small number of adjustable parameters. To begin with, the lattice spacing itself is not an observable. We are using the lattice to define the theory, and thus for physics we must take the continuum limit $a \to 0$. Then there is the coupling constant, which is also not a physical parameter due to the phenomenon of asymptotic freedom. The lattice works directly with a bare coupling, and in the continuum limit this should vanish as predicted by asymptotic freedom

$$g_0^2 \sim \frac{1}{\log(1/\Lambda a)} \to 0. \tag{3}$$

In the process, the coupling is replaced by an overall scale Λ, which can be regarded as an integration constant for the renormalization group equation. Coleman and Weinberg[27] gave this phenomenon of replacing a dimensionless coupling with a scale the marvelous name "dimensional transmutation." An overall scale is not really something we should expect to calculate from first principles. Its numerical value would depend on the units chosen, be they furlongs or light-fortnights.

Next consider the quark masses. These also renormalize to zero as a power of the coupling in the continuum limit. Factoring out this divergence, we can define a renormalized quark mass, a second integration constant of the renormalization group equations. One such constant M_i is needed for each quark "flavor" or species i. Up to an irrelevant overall scale, the physical theory is then a function only of the dimensionless ratios M_i/Λ. These are the only free parameters in the strong interactions. The origin of the underlying masses remains one of the outstanding mysteries of particle physics.

With multiple flavors, the massless quark limit gives a rather remarkable theory, one with no undetermined dimensionless parameters. This limit is not terribly far from reality; chiral symmetry breaking should give massless pions, and experimentally the pions are considerably lighter than the next hadron, the rho. A theory of two massless quarks is a fair approximation to the strong interactions at intermediate energies. In this limit all dimensionless ratios should be calculable from first principles, including quantities such as the rho to nucleon mass ratio.

Since it is absorbed into an overall scale, the strong coupling constant at any physical scale is not an input parameter, but should be determined from first principles. Such calculations have placed lattice gauge theory into the famous particle data group tables[28].

9. Numerical simulation

While other techniques exist, such as strong coupling expansions, large scale numerical simulations currently dominate the practice of lattice gauge theory. They are based on evaluating the path integral

$$Z = \int (dU)\ e^{-\beta S} \tag{4}$$

with β proportional to the inverse bare coupling squared. A direct evaluation of such an integral has pitfalls. At first sight, the basic size of the calculation is overwhelming. Considering a 10^4 lattice, small by today's standards, there are 40,000 links. On each is an $SU(3)$ matrix, parameterized by 8 numbers. Thus we have a $10^4 \times 4 \times 8 = 320,000$ dimensional integral. One might try to replace this with a discrete sum over values of the integrand. If we make the extreme approximation of using only two points per dimension, this gives a sum with

$$2^{320,000} = 3.8 \times 10^{96,329} \tag{5}$$

terms! Of course, computers are getting pretty fast, but one should remember that the age of universe is only $\sim 10^{27}$ nanoseconds.

These huge numbers suggest a statistical treatment. The above integral is formally a partition function. Consider a more familiar statistical system, such as a glass of beer. There are a huge number of ways of arranging the atoms of carbon, hydrogen, oxygen, etc. that still leave us with a glass of beer. We don't need to know all those arrangements, we only need a dozen or so "typical" glasses to know all the important properties.

This is the basis of the Monte Carlo approach. The analogy with a partition function and the role of $\frac{1}{\beta}$ as a temperature enables the use of standard techniques to obtain "typical" equilibrium configurations, where the probability of any given configuration is given by the Boltzmann weight

$$P(C) \sim e^{-\beta S(C)}. \tag{6}$$

For this we use a Markov process, making changes in the current configuration

$$C \to C' \to \ldots \tag{7}$$

biased by the desired weight.

The idea is easily demonstrated with the example of Z_2 lattice gauge theory.[29] For this toy model, the links are allowed to take only two values, either plus or minus unity. One sets up a loop over the lattice variables. When looking at a particular link, calculate the probability for it to have value 1

$$P(1) = \frac{e^{-\beta S(1)}}{e^{-\beta S(1)} + e^{-\beta S(-1)}}. \tag{8}$$

Then pull out a roulette wheel and select either 1 or -1 biased by this weight. Lattice gauge Monte-Carlo programs are by nature quite simple. They are basically a set of nested loops surrounding a random change of the fundamental variables.

Extending this to fields in larger manifolds, such as the $SU(3)$ matrices representing the gluon fields, is straightforward. The algorithms are usually based on a detailed balance condition for a local change of fields taking configuration C to configuration C'. If probabilities for making these changes in one step satisfy

$$\frac{P(C \to C')}{P(C' \to C)} = \frac{e^{-\beta S(C')}}{e^{-\beta S(C)}}, \tag{9}$$

a straightforward argument shows that any ensemble of configurations will approach the equilibrium ensemble.

The results of these simulations have been fantastic, giving first principles calculations for interacting quantum field theories. I will just mention a few examples. The early result that bolstered the lattice into mainstream particle physics was the convincing demonstration of the confinement phenomenon. The force between two quark sources indeed remains constant at large distances.

A major goal of lattice simulations is to understand the hadronic spectrum. This is done by studying the long distance behavior of correlation functions. Let $\phi(t)$ be an operator that can create a specific particle at time t. Then as t becomes large the correlator

$$\langle \phi(t)\phi(0)\rangle \longrightarrow e^{-mt} \tag{10}$$

where m is the mass of the lightest hadron that can be created by ϕ. In these calculations the bare quark masses are parameters that can be determined by fitting a few of the light mesons. Chiral symmetry is useful here, with the pion mass squared predicted to be proportional to the light quark mass. Using the pion mass to fix the light quark mass and the kaon mass to fix the strange quark, all other particle masses should be determined. In this way, recent simulations with physical mass pions have successfully mapped out much of the low energy hadron spectrum.[26].

Another accomplishment for which the lattice excels over all other methods has been the study (using an approximation to QCD) of the deconfinement of quarks and gluons into a plasma at a temperature of about 170–190 MeV[30]. Indeed, the lattice is a unique quantitative tool capable of making precise predictions for the value this temperature. The method is based on the fact that the Euclidean path integral in a finite temporal box directly gives the physical finite temperature partition function, where the size of the box is proportional to the inverse temperature. This transition represents the confining flux tubes becoming lost in a background plasma of virtual flux lines.

10. Concluding remarks

In summary, lattice gauge theory provides the dominant framework for investigating non-perturbative phenomena in quantum field theory. The approach is currently dominated by numerical simulations, although the basic framework is potentially considerably more general. With the recent developments towards implementing

chiral symmetry on the lattice, including domain-wall fermions, the overlap formula, and variants on the Ginsparg-Wilson relation, parity conserving theories, such as the strong interactions, are fundamentally in quite good shape.

A particularly fascinating unsolved issue is the chiral gauge problem. Without a proper lattice formulation of a chiral gauge theory, it is unclear whether such models make any sense as a fundamental field theories. This is important for understanding how neutrinos can couple in only one helicity state. A marvelous goal would be a fully finite, gauge invariant, and local lattice formulation of the standard model. The problems encountered with chiral gauge theory are closely related to similar issues with super-symmetry, another area that does not naturally fit on the lattice. Understanding these issues will be necessary to make ties with the explosive activity in string theory and a possible regularization of gravity.

The other major unsolved problems in lattice gauge theory are algorithmic. Current fermion algorithms are extremely awkward and computer intensive. It is unclear why this has to be so, and may only be a consequence of our working directly with fermion determinants. One could to this for bosons too, but that would clearly be terribly inefficient. At present, the fermion problem seems completely intractable when the fermion determinant is not positive. This is of more than academic interest since interesting superconducting phases are predicted at high quark density. Similar sign problems appear in other fields of physics, such as doped strongly coupled electron systems.

Finally, throughout history the question of "what is elementary?" continues to arise. This is almost certainly an ill posed question, with variant approaches being simpler in distinct contexts. At a more mundane level, for low energy chiral dynamics we lose nothing by considering the pion as an elementary pseudo-goldstone field, while at extremely short distances string structures may become more fundamental. Quarks and their confinement may only be a useful temporary construct along the way.

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