The Standard Model and the Lattice

Michael Creutz

Senior Physicist Emeritus

Physics Department

Brookhaven National Laboratory

Upton, NY 11973, USA *

Abstract

The $SU(3) \otimes SU(2) \otimes U(1)$ standard model maps smoothly onto a conventional lattice gauge formulation, including the parity violation of the weak interactions. The formulation makes use of the pseudo-reality of the weak group and requires the inclusion a full generation of both leptons and quarks. As in continuum discussions, chiral eigenstates of the Dirac operator generate known anomalies, although with rough gauge configurations these are no longer exact zero modes of the Dirac operator.

*Electronic address: `mike@latticeguy.net`

Lattice gauge theory has a long history of successes in the study of low energy QCD, the underlying theory of the strong nuclear force. One might ask if the approach could also be used for the weak interactions related to beta decay. This is probably of little value for calculations since the electroweak coupling is small and conventional perturbation theory is highly accurate for most purposes. However, from a theoretical point of view a lattice theory provides a path towards a mathematical definition of a field theory in the limit where the lattice spacing is taken to zero. Putting the weak interactions on the lattice is a first step toward a formal definition of the theory. Within the picture, gauge field topology gives rise to known anomalies and the weak interactions generate baryon decay through the effective vertex elucidated by 't Hooft [1]. A crucial ingredient here is the need to include entire generations to properly cancel anomalies. Although the approach can account for the parity violation in weak decays, the inclusion of electromagnetism and a non-trivial Higgs potential require non-asymptotically free couplings. Thus the path to a rigorous definition of the continuum limit remains elusive.

To remain as close to traditional lattice methods [2–4] as possible, consider all gauge fields as group elements on the bonds of a four dimensional hypercubic lattice. This includes elements of $SU(3)$ for the strong interactions, $SU(2)$ for the weak isospin group and $U(1)$ for hyper-charge, reflecting the fields of the usual standard model [5]. Denote these bond variables correspondingly as U_{su3}, U_{su2}, U_y. These fields self-interact through the standard gauge invariant plaquette form, although nothing here precludes improvement schemes. The goal is to maintain an exact local gauge symmetry under all three of these groups.

For one generation, include eight fermion fields, represented by

$$u^r, u^g, u^b, d^r, d^b, d^g, \nu, e^-. \tag{1}$$

The three colors $\{r, g, b\}$ are explicit for the up and down quarks u, d. Included also are the neutrino field ν and the electron e^-. These are all anticommuting Grassmann variables located on the lattice sites. In addition on each site are independent conjugate Grassmann variables

$$\overline{u^r}, \overline{u^g}, \overline{u^b}, \overline{d^r}, \overline{d^b}, \overline{d^g}, \overline{\nu}, \overline{e^-}. \tag{2}$$

With more generations this pattern is repeated for each.

All fermion fields are four component Dirac spinors. These can be divided into right and left handed parts

$$\psi_L = (1 - \gamma_5)\psi/2$$

$$\begin{aligned}\psi_R &= (1+\gamma_5)\psi/2 \\ \overline{\psi}_L &= \overline{\psi}(1+\gamma_5)/2 \\ \overline{\psi}_R &= \overline{\psi}(1-\gamma_5)/2.\end{aligned} \tag{3}$$

The weak interactions only couple directly to the left handed parts while the strong group and hyper-charge see both chiralities. Within the single generation of fermions, the three gauge groups behave rather differently. For the strong interactions there are two vector-like triplets under $SU(3)$

$$u = \begin{pmatrix} u^r \\ u^g \\ u^b \end{pmatrix}, \qquad d = \begin{pmatrix} d^r \\ d^g \\ d^b \end{pmatrix}. \tag{4}$$

In contrast, under the weak interactions there are four left handed doublets

$$r = \begin{pmatrix} u^r \\ d^r \end{pmatrix}_L, \qquad g = \begin{pmatrix} u^g \\ d^g \end{pmatrix}_L, \qquad b = \begin{pmatrix} u^b \\ d^b \end{pmatrix}_L, \qquad l = \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L. \tag{5}$$

Here the doublets are labeled by their color or lepton nature. Finally hyper-charges for the 8 fermions are taken as conventional in the standard model, differing between the left and right handed fermions. As listed in Eq. (1) for the left handed parts the assignments are

$$Y_L = (1/3, 1/3, 1/3, 1/3, 1/3, 1/3, -1, -1), \tag{6}$$

and for the right handed components

$$Y_R = (4/3, 4/3, 4/3, -2/3, -2/3, -2/3, 0, -2). \tag{7}$$

For the right handed fields these values are twice the electromagnetic charge while for the left handed parts they differ from twice the physical charges by ± 1. All gauge fields are neutral under hyper-charge.

The local gauge symmetries correspond to rotations of the various fields on the lattice sites. The strong group acts on the two quark triplets

$$\psi_{ud} \to g_{su3}\psi_{ud}. \tag{8}$$

Meanwhile the weak group acts on left handed doublets but leaves the right hand fields untouched

$$\psi_{rgbl} \to \left(g_{su2}\frac{1-\gamma_5}{2} + \frac{1+\gamma_5}{2} \right) \psi_{rgbl}. \tag{9}$$

In addition to the fermions, on each site is a complex doublet Higgs field

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}. \tag{10}$$

Both components of this field have hypercharge $Y = 1$. This field also rotates under the weak isospin

$$H \to g_{su2} H \tag{11}$$

but does not see the strong group.

The group $SU(2)$ is pseudo-real in the sense that a unitary transformation relates the complex conjugate of any element to itself. With the usual conventions

$$g^* = \tau_2 g \tau_2. \tag{12}$$

This means that in addition to the initial Higgs doublet there is another combination

$$H' \equiv \tau_2 H^* \tau_2 = \begin{pmatrix} -H_2^* \\ H_1^* \end{pmatrix} \tag{13}$$

that transforms equivalently

$$H' \to g_{su2} H' \tag{14}$$

with charge $Y = -1$. Finally the hyper-charge gauge group rotates the phases of all fields by an angle proportional to their respective hyper-charges

$$\psi \to e^{i\theta Y_\psi} \psi. \tag{15}$$

Note that these three gauge groups commute with each other. The weak group does not change the colors of the quarks. The strong group doesn't break weak isospin. And the hyper-charge assignments are constant within the strong and weak multiplets.

The appearance of an even number of fundamental weak doublets is essential. Witten [6] has discussed how a closed path in $SU(2)$ field space can change the sign of the fermion determinant. This is closely related to the 't Hooft vertex [1] connection to spin flips from the anomaly in vector multiplets. With only a left hand multiplet there is nothing to flip into. With an even number of multiplets this can be compensated among them, as discussed later.

The couplings of the fields to the various gauge bosons are contained in the standard hopping terms between sites, including such terms for the Higgs field. All left handed weak doublets rotate by the U_{su2} while the right handed counterparts do not see that matrix. The leptons do not interact with the U_{su3} while the quarks do. And all fermions rotate appropriately under hyper-charge.

To complete the picture requires the Higgs mechanism [7–11]. This will serve two purposes, generating masses and eliminating doublers associated with naive fermion hopping. The Higgs mass itself can be adjusted via a self interaction potential $V(|H|)$, as in the continuum treatment of the standard model. A quartic term is necessary in V in order to adjust the expectation value $v = \langle |H| \rangle$ to map directly onto conventional continuum phenomenology.

Concentrating on the weak interactions, for each fermion doublet there are two distinct on site combinations that are invariant under the weak gauge group

$$
\begin{aligned}
H^\dagger \psi &= H_1^* \psi_1 + H_2^* \psi_2 \\
H'^\dagger \psi &= -H_2 \psi_1 + H_1 \psi_2 .
\end{aligned}
\tag{16}
$$

It is convenient to divide out the Higgs expectation v and think of the physical left handed particles as "composite" with physical charges Q

$$
\begin{array}{ccccc}
e_L & = & H^\dagger l/v & & Q = (Y_l - Y_H)/2 = -1 \\
\nu_L & = & H'^\dagger l/v & & Q = (Y_l + Y_H)/2 = 0 \\
u_{[rgb]_L} & = & H^\dagger [rgb]/v & & Q = (Y_{rgb} - Y_H)/2 = 2/3 \\
d_{[rgb]_L} & = & H'^\dagger [rgb]/v & & Q = (Y_{rgb} + Y_H)/2 = -1/3 .
\end{array}
\tag{17}
$$

This is similar to working perturbatively in the "unitary" gauge, although in proper lattice gauge spirit all gauges are integrated over. For each doublet one can now form two gauge singlet mass terms

$$
\begin{aligned}
&\overline{\psi}_R M H^\dagger \psi_L + h.c. \\
&\overline{\psi}_R M' H'^\dagger \psi_L + h.c.
\end{aligned}
\tag{18}
$$

. Here M and M' are arbitrary mass matrices and can include any inter-generational mixings.

The same mechanism that gives the fermions masses can now be adapted to eliminate doublers using a Wilson [3] like mechanism. First remove the $SU(2)$ dependence with the Higgs field as in the mass term. Then include the Wilson projection operator $(1 \pm \gamma_\mu)/2$ for fermions to hop to neighboring sites. Thus, for each doublet hopping from site i to $i + e_\mu$ add to the action

$$
\begin{aligned}
&\overline{\psi}_{R i+e_\mu} (1+\gamma_\mu) H^\dagger \psi_{L i}/2v + \overline{\psi}_{R i} (1-\gamma_\mu) H^\dagger \psi_{L i+e_\mu}/2v + h.c. \\
&\overline{\psi}_{R i+e_\mu} (1+\gamma_\mu) H'^\dagger \psi_{L i}/2v + \overline{\psi}_{R i} (1-\gamma_\mu) H'^\dagger \psi_{L i+e_\mu}/2v + h.c.
\end{aligned}
\tag{19}
$$

Here the appropriate strong and hyper-charge matrices are suppressed for notational simplicity. In a sense this term mimics an irrelevant operator proportional to $\overline{\psi}\partial^2\psi$ which moves all doubler masses to the cutoff scale. As with the usual Wilson procedure, this requires an additive mass renormalization. Because of this all masses need to be fine tuned. In this approach the smallness of neutrino masses is not natural.

Combining the Higgs field with the $SU(2)$ bond variables allows construction of gauge invariant operators to create the physical W and Z bosons. For example

$$W_\mu^+ \sim H'^\dagger_{i+e_\mu}, U_{su2\,i+e_\mu,i} H_i \tag{20}$$

has charge $Q = 1$ and represents the W^+ with associated spin in the bond direction. Similarly the W^- corresponds to

$$W^- \sim H^\dagger_{i+e_\mu} U_{su2\,i+e_\mu,i} H'_i. \tag{21}$$

In addition there are two neutral combinations

$$\begin{array}{l} H'^\dagger_{i+e_\mu} U_{su2\,i+e_\mu,i} H'_i \\ H^\dagger_{i+e_\mu} U_{su2\,i+e_\mu,i} H_i. \end{array} \tag{22}$$

While these generally mix, the second term appears in the action as the hopping term for the Higgs field. Remaining is an operator for the physical Z.

This basically completes the model, but it is instructive to consider how the usual quantum anomalies come into play. In continuum discussions they are related to topology in the gauge fields and the index theorem [12, 13] relating to zero modes in the Dirac operator. On the lattice, however, the space of allowed configurations is simply connected and does not support separate topological sectors absent some sort of smoothing condition [14]. However such restrictions destroy reflection positivity [15] and will interfere with any Hamiltonian formulation.

The formulation presented here preserves gamma five hermeticity for the Dirac operator

$$\gamma_5 D \gamma_5 = D^\dagger. \tag{23}$$

Indeed most lattice fermion prescriptions, with the exception of twisted mass [16], satisfy this. An immediate consequence is that on diagonalizing D all eigenvalues are either real or in complex conjugate pairs. (Since D is not a normal operator, consider either left or right eigenvalues for this discussion.)

If the gauge fields are sufficiently smooth, the index theorem does apply and modes of nontrivial chirality are possible. Since the space of fields is simply connected, there exists a path connecting a configuration with such chiral states to one without, as discussed in Ref. [17]. In terms of the eigenvalues of D, a complex conjugate pair joins on the real axis and splits apart as

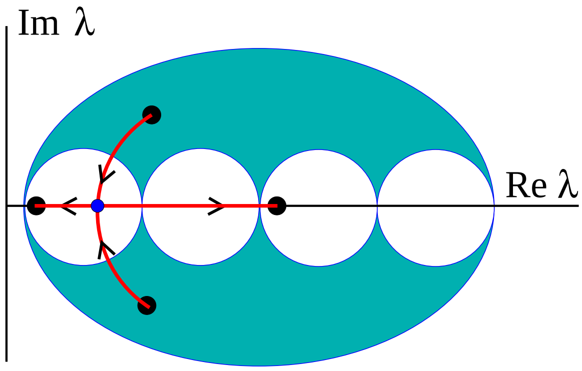

FIG. 1: *The merging of two complex eigenvalues to form two real ones, one of which can become small with the other moving into the doubler region. The background region represents the spectrum of free Wilson fermions. If the fields are smoothed into a classical instanton, the small eigenvalue becomes the zero mode from topology and the index theorem.*

two real eigenvalues. One can move to small real part while the other moves off to the doubler region, as sketched in Fig. 1.

Consider the space spanned by the real eigenvalues of D. On this sub-space γ_5 commutes with D and can be simultaneously diagonalized. Thus the states can be labeled by chirality. The usual topological structures are represented by an imbalance of small eigenvalues of one chirality over the other. If a gauge field configuration with such a mode is now smoothed, the small eigenvalue will be driven to zero and satisfy the continuum index theorem. As the full trace of γ_5 must vanish, such zero modes have corresponding modes of the opposite chirality in the doubler region.

These chiral eignmodes are directly tied to quantum anomalies as discussed by 't Hooft [1]. Small or zero eigenvalues suppress the partition function

$$Z = \int (dA)\ (d\overline{\psi} d\psi) e^{-S_g + \overline{\psi} D \psi} = \int (dA)\ e^{-S_g(A)} \prod \lambda_i. \tag{24}$$

On the surface, this suggests that zero modes are irrelevant as they don't appear in the partition function. But 't Hooft showed how certain observables can overcome this suppression. To see this, first introduce abstract sources η and $\overline{\eta}$

$$Z(\eta, \overline{\eta}) = \int (dA)\ (d\overline{\psi} d\psi)\ e^{-S_g + \overline{\psi} D \psi + \overline{\psi} \eta + \overline{\eta} \psi}. \tag{25}$$

Differentiation (in a Grassmann sense) with respect to the sources generates the Green's functions of the theory. Completing the square and doing the integral over the fermions gives

$$Z = \int (dA)\ e^{-S_g + \overline{\eta} D^{-1} \eta / 4} \prod \lambda_i. \tag{26}$$

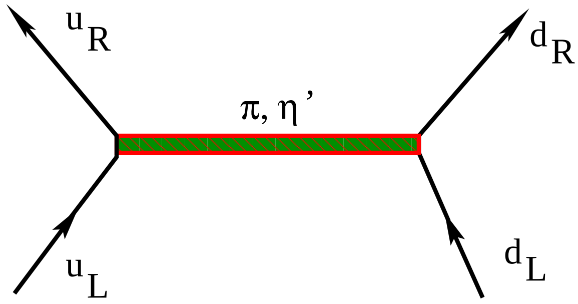

FIG. 2: *The chiral anomaly produces a spin flip amplitude involving all quark species. The non-vanishing of this diagram induces a mass for the eta prime and causes a mixing of the up and down quark masses when they are not degenerate. Figure taken from Ref. [18].*

If the sources overlap with one of the small eigenmodes, the inverse of the corresponding eigenvalue can enter the Green's function and cancel the suppression from the small eigenvalue in the partition function.

This effect is well understood for the strong interactions. A chiral eigenmode couples left handed quarks to right handed ones, resulting in a non-vanishing spin flip amplitude, even if the quarks are mass-less. The mixing of various pseudo-scalars through the process is sketched in Fig. 2.

The consequences of the anomaly for the weak interactions are less familiar, primarily since the effect is quite small. The non-perturbative chiral modes will be suppressed exponentially in the inverse electroweak coupling. Although tiny, the effects are crucial to understanding the structure of the theory. For each left handed $SU(2)$ doublet, its conjugate field is right handed

$$\psi^c = \tau_2 \gamma_2 \psi^*. \tag{27}$$

For example, the anti-neutrino is right handed. Of our four doublets $\{r,g,b,l\}$, take two and pair them with the conjugates of the other two and then antisymmetrize over the combinations. Using sources that overlap with a low chiral mode gives a non-vanishing value for the four point "vertex"

$$\varepsilon_{ijkl} \langle \overline{\psi}_i^c \overline{\psi}_j^c D^{-1} \psi_k \psi_l \rangle \neq 0. \tag{28}$$

Here the indices run over the four doublets $\{r,g,b,l\}$.

This effective interaction violates both baryon and lepton number, but preserves the difference $B-L$. While this discussion is for the euclidean formulation of the theory, in a Hamiltonian approach the process proceeds from modes crossing in and out of the Dirac sea. In the process, fermion number changes by 2 units. This is consistent with $SU(2)$ since the group is pseudo-real. It is also consistent with the $SU(3)$ symmetry since $\overline{3} \in 3 \otimes 3$ and two flavors are going to one anti-flavor. A version of this vertex appears in a proposed domain wall approach to the weak interactions [19–21]. The net process can be thought of as an effective mixing of the anti-neutron with the neutrino and the anti-proton with the electron

$$\begin{pmatrix} n^c \\ p^c \end{pmatrix}_R \Longleftrightarrow \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L . \tag{29}$$

Through this mechanism proton decay $p \to e^+ + \pi$ is allowed, although it is extremely small being suppressed exponentially in $1/\alpha$.

In summary, one generation of the standard model fits nicely into a conventional lattice gauge framework. The approach keeps the $SU(3) \otimes SU(2) \otimes U(1)$ gauge symmetries exact. To be consistent with known anomalies, the approach requires including each generation in its entirety. The small baryon and lepton violation demonstrated by 't Hooft appears in the lattice approach through the appearance of chiral eigenmodes of the Dirac operator. Unlike in the continuum where differentiable fields are assumed, these modes are not forced to occur exactly at zero. The overall picture is close in spirit to confining models [22–24]. The main remaining issue concerns asymptotic freedom, absent both in electromagnetism and the Higgs quartic self coupling. This leaves an obstacle towards defining the theory in the continuum limit. For electromagnetism this suggests a possible unification with further gauge fields at high energies. For the Higgs it hints at composite models or possibly involving gravity at the highest energies [25, 26].

Acknowledgments

This manuscript has been authored under contract number DE-AC02-98CH10886 with the U.S. Department of Energy. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.

[1] G. 't Hooft, Phys.Rev. **D14**, 3432 (1976).

[2] K. G. Wilson, Phys. Rev. **D10**, 2445 (1974).

[3] K. G. Wilson, Erice Lectures 1975 (1977), new Phenomena In Subnuclear Physics. Part A. Proceedings of the First Half of the 1975 International School of Subnuclear Physics, Erice, Sicily, July 11 - August 1, 1975, ed. A. Zichichi, Plenum Press, New York, 1977, p. 69, CLNS-321.

[4] M. Creutz, *Quarks, Gluons and Lattices*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2023).

[5] R. Oerter, *The theory of almost everything: The standard model, the unsung triumph of modern physics* (2006).

[6] E. Witten, Phys. Lett. B **117**, 324 (1982).

[7] P. W. Higgs, Phys. Rev. Lett. **13**, 508 (1964), URL `https://link.aps.org/doi/10.1103/PhysRevLett.13.508`.

[8] S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967).

[9] A. Salam, Conf. Proc. **C680519**, 367 (1968).

[10] F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964), URL `https://link.aps.org/doi/10.1103/PhysRevLett.13.321`.

[11] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Lett. **13**, 585 (1964), URL `https://link.aps.org/doi/10.1103/PhysRevLett.13.585`.

[12] M. Atiyah and I. Singer, Bull.Am.Math.Soc. **69**, 422 (1969).

[13] M. Atiyah and I. Singer, Annals Math. **93**, 139 (1971).

[14] M. Luscher, Commun.Math.Phys. **85**, 39 (1982).

[15] M. Creutz, Phys. Rev. **D70**, 091501 (2004), hep-lat/0409017.

[16] R. Frezzotti, P. A. Grassi, S. Sint, and P. Weisz (Alpha collaboration), JHEP **0108**, 058 (2001), hep-lat/0101001.

[17] M. Creutz, Nucl. Phys. Proc. Suppl. **119**, 837 (2003), hep-lat/0208026.

[18] M. Creutz, Phys. Rev. **D83**, 016005 (2011), 1010.4467.

[19] M. Creutz, M. Tytgat, C. Rebbi, and S.-S. Xue, Phys. Lett. **B402**, 341 (1997), hep-lat/9612017.

[20] M. Creutz, Nucl. Phys. Proc. Suppl. **63**, 599 (1998), hep-lat/9708020.

[21] M. Creutz, *From Quarks to Pions* (WSP, 2018), URL `https://www.worldscientific.com/worldscibooks/10.1142/10688#t=toc`.

[22] K. Osterwalder and E. Seiler, Annals Phys. **110**, 440 (1978).

[23] E. H. Fradkin and S. H. Shenker, Phys. Rev. **D19**, 3682 (1979).

[24] J. Greensite and K. Matsuyama, Phys. Rev. D **96**, 094510 (2017), 1708.08979.

[25] M. Shaposhnikov and C. Wetterich, Phys. Lett. B **683**, 196 (2010), 0912.0208.

[26] S. Alekhin, A. Djouadi, and S. Moch, Phys. Lett. B **716**, 214 (2012), 1207.0980.