Minimally doubled chiral fermions

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Chiral symmetry crucial to our understanding of hadronic physics

- pions are waves on a background quark condensate $\langle\overline{\psi}\psi\rangle$
- chiral extrapolations essential to practical lattice calculations

Anomaly removes classical $U(1)$ chiral symmetry

- $SU(N_f) \times SU(N_f) \times U_B(1)$
- non trivial symmetry requires $N_f \geq 2$

On the lattice ignoring the anomaly gives doublers

- naive fermions: 16 species, exact $U(4)_L \times U(4)_R$ symmetry
- staggered fermions: 4 species (tastes), one exact chiral symmetry
- Wilson fermions: one light species
  - all chiral symmetries broken by doubler mass term
- overlap, domain wall, perfect actions: N_f arbitrary but
  - not ultra-local: computationally intensive
  - anomaly hidden, $\gamma_5 \neq \hat{\gamma}_5$, $\mathrm{Tr}\hat{\gamma}_5 = 2\nu \neq 0$

Minimally doubled chiral fermion actions have just 2 species

- Karsten 1981
- Wilczek 1987
- recent revival: MC, Borici, Bedaque Buchoff Tiburzi Walker-Loud

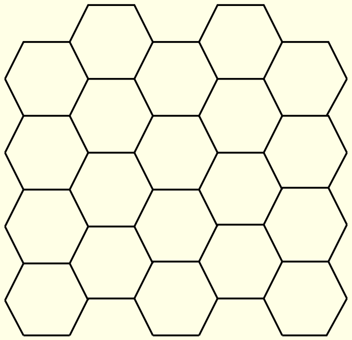
Motivations

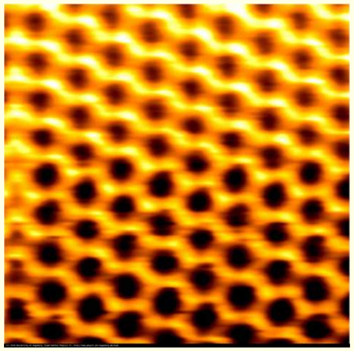
- failure of rooting for staggered
- lack of chiral symmetry for Wilson
- computational demands of overlap, domain-wall approaches

Elegant connection to the electronic structure of graphene

- vanishing mass protected by topological considerations

Graphene: two dimensional hexagonal lattice of carbon atoms





- http://online.kitp.ucsb.edu/online/bblunch/castroneto/
- A. H. Castro Neto et al., arXiv:0709.1163

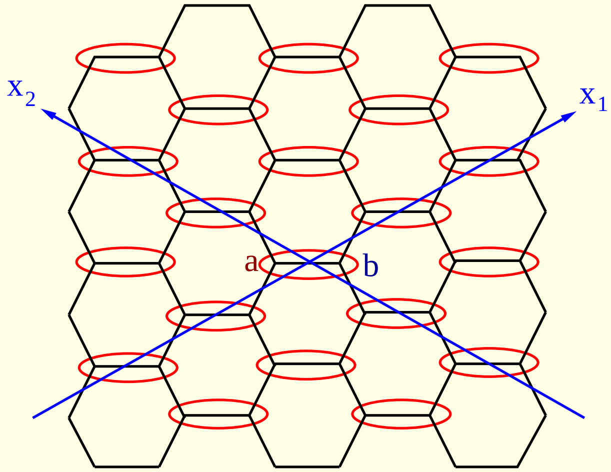
Held together by strong “sigma” bonds, sp^2

One “pi” electron per site can hop around

Consider only nearest neighbor hopping in the pi system

- tight binding approximation

Fortuitous choice of coordinates helps solve



Form horizontal bonds into ''sites'' involving two types of atom

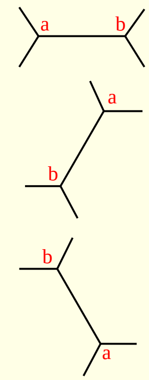
- ''a'' on the left end of a horizontal bond
- ''b'' on the right end
- all hoppings are between type a and type b atoms

Label sites by non-orthogonal coordinates x_1 and x_2

- axes at 30 degrees from horizontal

Hamiltonian

$$
\begin{aligned}
H = K \sum_{x_1,x_2} & a^\dagger_{x_1,x_2} b_{x_1,x_2} + b^\dagger_{x_1,x_2} a_{x_1,x_2} \\
& + a^\dagger_{x_1+1,x_2} b_{x_1,x_2} + b^\dagger_{x_1-1,x_2} a_{x_1,x_2} \\
& + a^\dagger_{x_1,x_2-1} b_{x_1,x_2} + b^\dagger_{x_1,x_2+1} a_{x_1,x_2}
\end{aligned}
$$



- hops always between a and b sites

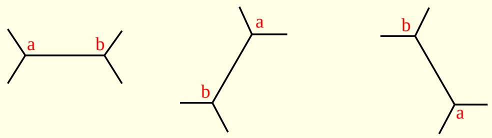
Go to momentum (reciprocal) space

- $a_{x_1,x_2} = \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \; e^{ip_1 x_1} \; e^{ip_2 x_2} \; \tilde{a}_{p_1,p_2}.$
- $-\pi < p_\mu \le \pi$

Hamiltonian breaks into two by two blocks

$$H = K \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \; (\, \tilde{a}^\dagger_{p_1,p_2} \quad \tilde{b}^\dagger_{p_1,p_2} \,) \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \begin{pmatrix} \tilde{a}_{p_1,p_2} \\ \tilde{b}_{p_1,p_2} \end{pmatrix}$$

- where $z = 1 + e^{-ip_1} + e^{+ip_2}$



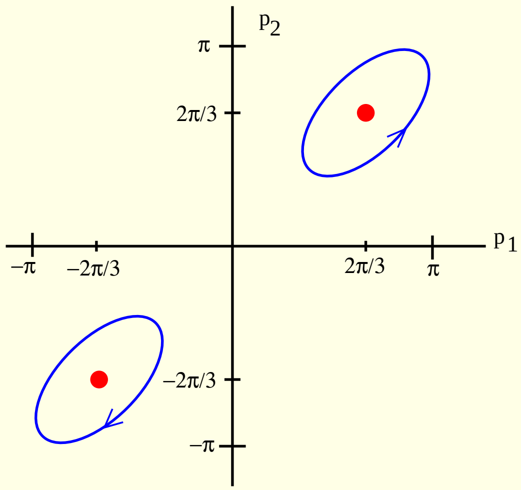
$$\tilde{H}(p_1, p_2) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$$

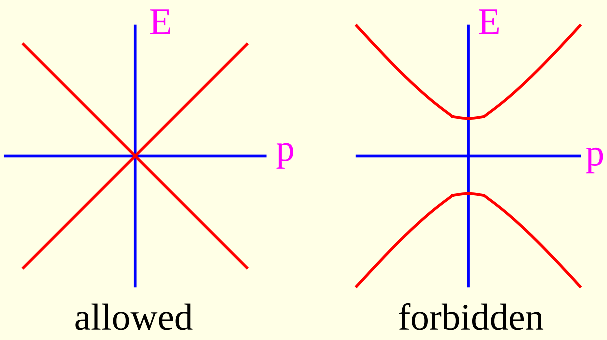
Fermion energy levels at $E(p_1, p_2) = \pm K|z|$

- energy vanishes only when $|z|$ does
- exactly two points $p_1 = p_2 = \pm 2\pi/3$

Topological stability

- contour of constant energy near a zero point
- phase of z wraps around unit circle
- cannot collapse contour without going to $|z| = 0$





No band gap allowed

- Graphite is black and a conductor

No-go theorem

Nielsen and Ninomiya

- periodicity of Brillouin zone
- wrapping around one zero must unwrap elsewhere
- two zeros is the minimum possible

Connection with chiral symmetry

- $b \longrightarrow -b$ changes sign of H
- $\tilde{H}(p_1, p_2) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$ anticommutes with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  - $\sigma_3 \longrightarrow \gamma_5$ in four dimensions

Four dimensions

Want Dirac operator D to put into path integral action $\overline{\psi} D \psi$

- require "γ_5 Hermiticity"
  - $\gamma_5 D \gamma_5 = D^\dagger$
- work with Hermitean "Hamiltonian" $H = \gamma_5 D$
  - not the Hamiltonian of the 3D Minkowski theory

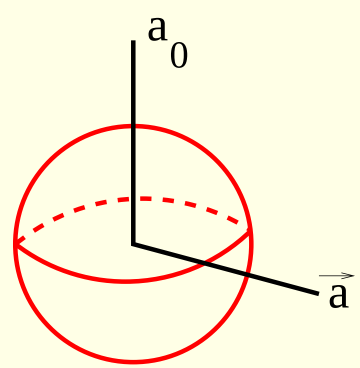
Require same form as the two dimensional case

$$\tilde{H}(p_\mu) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$$

- four component momentum, (p_1, p_2, p_3, p_4)

To keep topological argument

- extend z to quaternions
- $z = a_0 + i\vec{a} \cdot \vec{\sigma}$
    - $|z|^2 = \sum_\mu a_\mu^2$



$\tilde{H}(p_\mu)$ now a four by four matrix

- "energy" eigenvalues still $E(p_\mu) = \pm K|z|$
- constant energy surface topologically an S_3
    - surrounding a zero should give non-trivial mapping

Implementation

- not unique
- here I follow Borici's construction

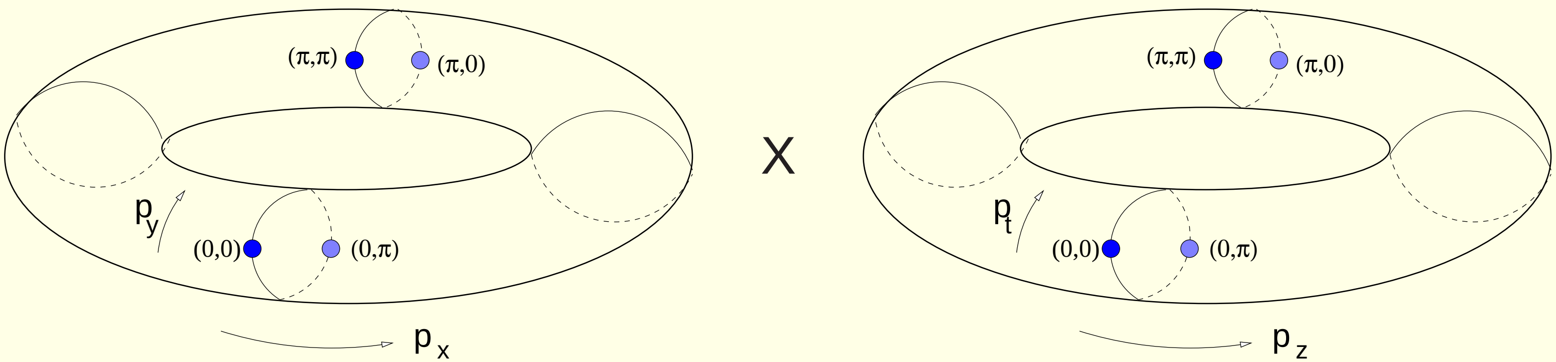
Start with naive fermions

- forward hop between sites $\gamma_\mu U$ — unit hopping parameter for convenience
- backward hop between sites $-\gamma_\mu U^\dagger$
  - μ is the direction of the hop
  - U is the usual gauge field matrix
- Dirac operator D anticommutes with γ_5
  - an exact chiral symmetry
  - part of an exact $SU(4) \times SU(4)$ chiral algebra — Karsten and Smit

In the free limit, solution in momentum space

$$D(p) = 2i \sum_\mu \gamma_\mu \sin(p_\mu)$$

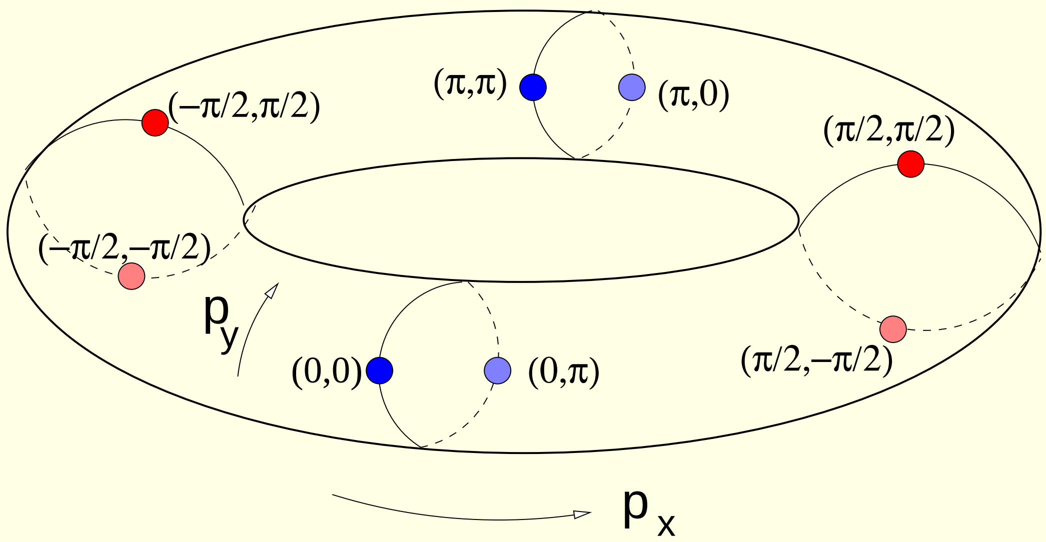
- for small momenta reduces to Dirac equation
- 15 extra Dirac equations for components of momenta near 0 or π



16 ‘‘Fermi points’’

- ‘‘doublers’’

Consider momenta maximally distant from the zeros: $p_\mu = \pm\pi/2$



Select one of these points, i.e. $p_\mu = +\pi/2$ for every μ

- $D(p_\mu = \pi/2) = 2i\sum_\mu \gamma_\mu \equiv 4i\Gamma$
- $\Gamma \equiv \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$
  - unitary, Hermitean, traceless 4 by 4 matrix

Now consider a unitary transformation

- $\psi'(x) = e^{-i\pi(x_1+x_2+x_3+x_4)/2}\ \Gamma\ \psi(x)$
- $\overline{\psi}'(x) = e^{i\pi(x_1+x_2+x_3+x_4)/2}\ \overline{\psi}(x)\ \Gamma$

- phases move Fermi points from $p_\mu \in \{0, \pi\}$ to $p_\mu \in \{\pm\pi/2\}$
- ψ' uses new gamma matrices $\gamma'_\mu = \Gamma\gamma_\mu\Gamma$
  - $\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) = \Gamma'$
- new free action: $\overline{D}(p) = 2i\sum_\mu \gamma'_\mu \sin(\pi/2 - p_\mu)$

D and $\overline{D}$ physically equivalent

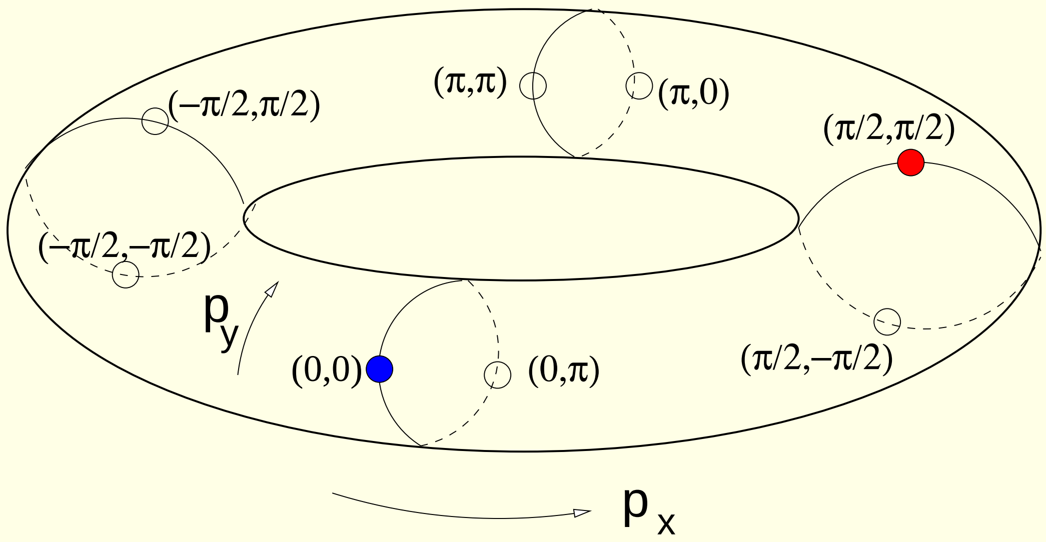
Complimentarity: $D(p_\mu = \pi/2) = \overline{D}(p_\mu = 0) = 4i\Gamma$

Combine the naive actions

$$\boxed{\mathcal{D} = D + \overline{D} - 4i\Gamma}$$

Free theory

- $\mathcal{D}(p) = 2i\sum_\mu \left(\gamma_\mu \sin(p_\mu) + \gamma'_\mu \sin(\pi/2 - p_\mu)\right) - 4i\Gamma$
- at $p_\mu \sim 0$ the $4i\Gamma$ term cancels $\overline{D}$, leaving $\mathcal{D}(p) \sim \gamma_\mu p_\mu$
- at $p_\mu \sim \pi/2$ the $4i\Gamma$ term cancels D, leaving $\mathcal{D}(\pi/2 - p) \sim \gamma'_\mu p_\mu$
  - Only these two zeros of $\mathcal{D}(p)$ remain!



THEOREM: these are the only zeros of $\mathcal{D}(p)$

- at other zeros of D, $\overline{D} - 4i\Gamma$ is large
- at other zeros of $\overline{D}$, $D - 4i\Gamma$ is large

Chiral symmetry remains exact

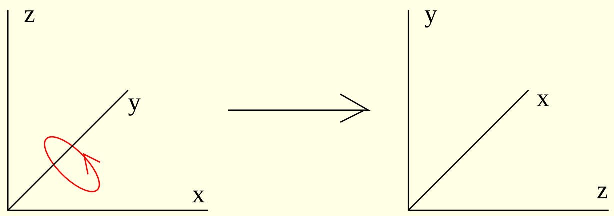
- $\gamma_5 \mathcal{D} = -\mathcal{D}\gamma_5$
- $e^{i\theta\gamma_5}\mathcal{D}e^{i\theta\gamma_5} = \mathcal{D}$

But

- $\gamma_5' = \Gamma\gamma_5\Gamma = -\gamma_5$
- two species rotate oppositely
- symmetry is flavor non-singlet

Space time symmetries

- usual discrete translation symmetry
- $\Gamma = \frac{1}{2}\sum_\mu \gamma_\mu$ treats primary hypercube diagonal specially
- action symmetric under subgroup of the hypercubic group
  - leaving this diagonal invariant
- includes Z_3 rotations amongst any three positive directions
  - $V = \exp((i\pi/3)(\sigma_{12} + \sigma_{23} + \sigma_{31})/\sqrt{3})$ $\quad [\gamma_\mu, \gamma_\nu] = 2i\sigma_{\mu\nu}$
  - cyclicly permutes x_1, x_2, x_3 axes $\quad [V, \Gamma] = 0$
  - physical rotation by $2\pi/3$



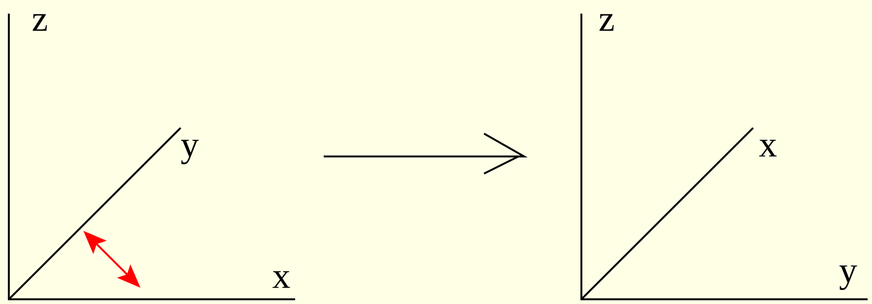
  - $V^3 = -1$: we are dealing with fermions

Repeating with other axes generates the 12 element tetrahedral group

- subgroup of the full hypercubic group

Odd-parity transformations double the symmetry group to 24 elements

- $V = \frac{1}{2\sqrt{2}}(1 + i\sigma_{15})(1 + i\sigma_{21})(1 + i\sigma_{52})$ $[V, \Gamma] = 0$
- permutes x_1, x_2 axes
- $\gamma_5 \to V^\dagger \gamma_5 V = -\gamma_5$



Natural time axis along main diagonal $e_1 + e_2 + e_3 + e_4$

- T exchanges the two Fermi points
- increases symmetry group to 48 elements

Karsten and Wilczek actions

- e_4 as the special direction

Charge conjugation: equivalent to particle hole symmetry

- $\mathcal{D}$ and $\mathcal{H} = \gamma_5 \mathcal{D}$ have eigenvalues in opposite sign pairs

Special treatment of main diagonal

- interactions can induce lattice distortions along this direction
  - $\frac{1}{a}(\cos(ap) - 1)\overline{\psi}\Gamma\psi = O(a)$
  - symmetry restored in continuum limit
- at finite lattice spacing can tune — Bedaque Buchoff Tiburzi Walker-Loud
  - coefficient of $i\overline{\psi}\Gamma\psi$ — dimension 3 operator
  - 6 link plaquettes orthogonal to this diagonal
- zeros topologically robust under such distortions
  - Nielsen Ninomiya, MC

Issues and questions

Requires a multiple of two flavors

- can split degeneracies with Wilson terms

Only one exact chiral symmetry

- not the full $SU(2) \otimes SU(2)$
  - enough to protect mass
  - π^0 a Goldstone boson
  - $\pi^{\pm}$ only approximate

Not unique

- only need $z(p)$ with two zeros
- above: Borici's variation with orthogonal coordinates
  - alternatives: Karsten, Wilczek, MC

Comparison with staggered

- both have one exact chiral symmetry
- both have only approximate zero modes from topology
- four component versus one component fermion field
- two versus four flavors
  - no uncontrolled extrapolation to two physical light flavors

Summary

- A strictly local lattice fermion action $\mathcal{D}(A)$
  - with one exact chiral symmetry $\gamma_5 \mathcal{D} = -\mathcal{D}\gamma_5$
  - describing two flavors; minimum required for chiral symmetry
  - a linear combination of two "naive" fermion actions (Borici)
- Space-time symmetries
  - translations plus 48 element subgroup of hypercubic rotations
  - includes odd parity transformations
  - renormalization can induce anisotropy at finite a