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MIGDAL-KADANOFF STUDY OF Z_4 SYMMETRIC SYSTEMS WITH GENERALIZED ACTION*

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For the Z_4 lattice gauge theory in four dimensions or the related spin system in two dimensions, we consider real space renormalization group transformations on the most general single plaquette or nearest neighbor action. The Migdal-Kadanoff recursion relations in differential form have several fixed points and predict a rich phase structure.

1. Introduction

The real space renormalization group is a powerful and elegant tool for the study of phase transitions in statistical systems. Concepts such as the universality of critical exponents are readily explained in terms of renormalization group flows in the vicinity of fixed points. In this paper we discuss how generalized Z_4 models provide a simple illustration of these ideas in a two-parameter coupling constant space.

For an exact treatment, one should consider renormalization group equations for an infinite number of coupling constants, involving both nearest and non-nearest neighbor interactions. For practical calculation some truncation to a few couplings is necessary. The Migdal-Kadanoff recursion relations provide such a truncation [1]. Although the errors involved are difficult to assess, the approximation is particularly simple and makes specific predictions for critical temperatures and exponents. For gauge theories the relations predict the critical nature of four dimensions. Indeed, before the advent of Monte Carlo calculations [2], this was the strongest evidence for quark confinement in the non-Abelian gauge theory of the strong interactions. Unfortunately, the recursion relations often make incorrect predictions on the order of the gauge theory transitions. Nevertheless, they appear to correctly predict when a phase transition is to be expected. This, plus the simplicity of the method, makes it worthy of further study.

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Recently considerable attention has been paid to alternatives to the Wilson action in lattice gauge theory [3]. One type of variant continues to keep the action as a sum over the lattice plaquettes but generalizes the class function associated with these plaquettes. As a continuous group has an infinite number of characters, the most general such action involves an infinite number of parameters. Again for practical calculations a truncation is necessary. Monte Carlo simulation for the SU(2) model with the action being a linear combination of fundamental and adjoint traces revealed a rich phase structure [4]. Much of this phase diagram was recently reproduced via the Migdal-Kadanoff relation although, as usual, the first order nature of the transitions did not appear [5].

In this paper we treat the considerably simpler group Z_4. With this group the most general plaquette action has only two dynamical parameters and no truncation of the character expansion is necessary. Having two parameters permits one to study renormalization group flows on a two-dimensional surface. We will see predicted critical behavior along lines in this plane which end at renormalization group fixed points. This simple example illustrates the universality of the critical exponents along these lines. Several features of the phase diagram bear a strong qualitative resemblance to the structures in the two coupling SU(2) model.

The Migdal-Kadanoff relations predict identical structure for spin systems in two dimensions and gauge theories in four [1]. Thus our discussion applies to both. The two-dimensional Z_4 nearest neighbor spin model is equivalent to the Ashkin-Teller model [6] which consists of two coupled Ising models. It is perhaps interesting to note that the four-dimensional Z_4 gauge model is not equivalent to two coupled Z_2 gauge models. Because the elements of Z_4 are not all their own inverses, the orientation of plaquettes plays a role in the strong coupling expansion. Strong coupling diagrams involving non-orientable and non-interacting closed surfaces (Klein bottles) differ between Z_4 and $Z_2 \times Z_2$ models. In less than four dimensions these diagrams do not play a role and the models are equivalent*.

The general Z_4 model also has the nice property of being self dual. The recursion relations in differential form preserve this symmetry and thus can exactly locate transition points on the self-dual line. In addition, certain dual pairs of critical lines are predicted.

The outline of this paper is as follows. In sect. 2 we review the general Migdal-Kadanoff approach and show that the resulting renormalization group equations respect duality for Z_p systems. In sect. 3 we specialize to Z_4 and find the explicit renormalization group function. Sect. 4 contains a discussion of the predicted fixed points and flows. The universality of the critical exponents along the transition lines is discussed. Conclusions and comments on certain missed features of the phase diagram appear in sect. 5.

* In the literature there is an incorrect claim on this point, ref. [7].

2. The Migdal-Kadanoff recursion relation

The Migdal-Kadanoff recursion relations represent a simple approximate method for comparing theories with different lattice spacings. We will discuss them for d-dimensional gauge theories, although the renormalization group functions thus found are identical up to a normalization factor to those of the $d/2$ dimensional nearest neighbor spin system.

We begin with the usual lattice gauge theory variables, elements $\{U_{ij}\}$ of the gauge group located on the bonds $\{ij\}$ of a hypercubic lattice. Associated with each plaquette $\square$ is the group element $U_\square$ representing the product of link variables around the plaquette with an arbitrary starting site. The partition function or path integral takes the form

$$Z = \int (\mathrm{d}U) \mathrm{e}^S, \tag{2.1}$$

where the action is a sum over all plaquettes

$$S = \sum_\square S_\square(U_\square) \tag{2.2}$$

and all links are integrated over the group. The action for any plaquette is a real class function

$$S_\square(U) = S_\square(gUg^{-1}) = S_\square^*(U), \tag{2.3}$$

when g is an arbitrary group element. We introduce two sets of parameters, one for the action and one for the exponentiated action, using the character expansion

$$S_\square(U) = \sum_n \beta_n \chi_n(U), \tag{2.4}$$

$$\mathrm{e}^{S_\square(U)} = \sum_n b_n \chi_n(U). \tag{2.5}$$

Here $\chi_n(U)$ is the trace of U in the nth irreducible representation of the gauge group. The characters satisfy the orthogonality relations

$$\sum_n \chi_n^*(U)\chi_n(U') = \delta(U, U'), \tag{2.6}$$

$$\int \mathrm{d}U \chi_n^*(U) \chi'_n(UU') = d_n^{-1} \delta_{nn'} \chi_n(U'), \tag{2.7}$$

where d_n is the dimension of the matrices in the nth representation, $\mathrm{d}U$ is the invariant group measure, and $\delta(U,U')$ is the invariant delta function on the group. These relations allow us to invert eqs. (2.4, 2.5)

$$\beta_n = \int \mathrm{d}U \chi_n^*(U) S_\square(U), \tag{2.8}$$

$$b_n = \int \mathrm{d}U \chi_n^*(U) \mathrm{e}^{S_\square(U)}. \tag{2.9}$$

We now wish to compare this theory to one on a lattice of larger spacing. We will first change the spacing along one direction and then repeat the process on the other directions. This process will introduce anisotropies in the couplings so we temporarily label the couplings with indices (μ,ν) denoting the planes of the plaquettes

$$S_\square(U) \to S^{(\mu,\nu)}(U), \tag{2.10}$$

$$\beta_n \to \beta_n^{(\mu,\nu)}, \tag{2.11}$$

$$b_n \to b_n^{(\mu,\nu)}. \tag{2.12}$$

Let us first consider doubling the spacing in say the x direction. We wish to integrate out all variables with odd x coordinate. To simplify matters it is useful to temporarily go into an axial gauge and set all links in the x direction to unity. Then plaquettes parallel to the x direction represent nearest neighbor couplings of the unfixed links at successive x coordinates. Plaquettes orthogonal to the x direction couple links all with the same x coordinate. On integrating out links with some given x coordinate, these orthogonal plaquettes will induce long range couplings. To eliminate these, the approximation of plaquette moving is made. All plaquettes with only odd x coordinates are neglected and, to compensate, those with even x coordinates have their couplings β increased by a factor of two. Thus the recursion takes orthogonal couplings to stronger values

$$\beta_n^{(\mu,\nu)} \to 2\beta_n^{(\mu,\nu)} \qquad \text{if } \mu \text{ and } \nu \neq x. \tag{2.13}$$

The integration over the odd sites is now simply a convolution of the plaquettes parallel to the x axis and gives the new action

$$\exp\left(S^{(x,\nu)}(U)\right) \to \int \mathrm{d}U' \, \mathrm{e}^{S^{(x,\nu)}(U')} \mathrm{e}^{S^{(x,\nu)}(U'^{-1}U)}. \tag{2.14}$$

In the Fourier transform space of the characters, convolutions are a simple product; the relations (2.6) and (2.7) give

$$b_n^{(x,\nu)} \to d_n^{-1}\left(b_n^{(x,\nu)}\right)^2. \tag{2.15}$$

One can now undo the gauge fixing, go to a new direction, and repeat the entire procedure.

Unfortunately these approximate decimations in the various directions do not commute. This gives rise to anisotropic couplings even if the initial ones were isotropic. The simplest way to avoid this is to do the successive decimations in an infinitesimal manner. The eqs. (2.13) and (2.15) are easily modified to a differential change in the x lattice spacing a_x and become

$$a_x \frac{\mathrm{d}}{\mathrm{d}a_x} \beta_n^{(\mu,\nu)} = \beta_n^{(\mu,\nu)} \qquad \text{if } \mu \text{ and } \nu \neq x, \tag{2.16}$$

$$a_x \frac{\mathrm{d}}{\mathrm{d}a_x} b_n^{(x,\nu)} = b_n^{(x,\nu)} \log\left(b_n^{(x,\nu)} / d_n \right). \tag{2.17}$$

We can now change the lattice spacings in all directions and drop the direction indices to obtain the final result for the renormalization group functions

$$a \frac{\mathrm{d}}{\mathrm{d}a} \beta_n = (d-2)\beta_n + 2 \sum_{n'} \frac{\partial \beta_n}{\partial b_{n'}} b_{n'} \log(b_{n'} / d_n). \tag{2.18}$$

The number 2 appears in this equation because each plaquette involves two of the d dimensions. For a spin theory these become ones.

We now wish to show that this recursion relation preserves the self duality of the four-dimensional model with gauge group

$$Z_p = \{ \exp(2\pi i n / p) | n = 0, \dots, p-1 \}. \tag{2.19}$$

This group has exactly p irreducible representations, all one dimensional, and given by

$$R_n(U) = U^n, \qquad n = 0, \dots, p-1. \tag{2.20}$$

Thus eqs. (2.4) and (2.5) are simply

$$S_\square(U) = \sum_{n=0}^{p-1} \beta_n U^n, \tag{2.21}$$

$$\mathrm{e}^{S_\square(U)} = \sum_{n=0}^{p-1} b_n U^n. \tag{2.22}$$

Duality is the statement [8,9]

$$Z(b) = Z(\tilde{b}), \tag{2.23}$$

where

$$\tilde{b}_n = A_{nm} b_m , \tag{2.24}$$

and the unitary matrix A is the discrete Fourier transform

$$A_{mn} = \frac{1}{\sqrt{p}} e^{2\pi i mn/p} . \tag{2.25}$$

Rewriting the recursion relation for the variables b_n in $d = 4$ gives

$$a \frac{d}{da} b_n = 2 b_n \log b_n + 2 \sum_{n'} \frac{\partial b_n}{\partial \beta_{n'}} \beta_{n'} . \tag{2.26}$$

A little algebra with the character orthogonality relation makes the second term simple in terms of the dual variables

$$a \frac{d}{da} b_n = 2 b_n \log b_n + 2 \sum_m A^*_{nm} \tilde{b}_m \log \tilde{b}_m + b_n \log p . \tag{2.27}$$

Multiplying this equation by the matrix A gives the same equation with b and $\tilde{b}$ interchanged and A replacing A^*. Thus we see that the operation of bond moving and decimation are dual to each other.

3. The Z_4 renormalization group equations

Specializing to the group $Z_4 = \{\pm 1, \pm i\}$, we consider the plaquette action

$$S_\Box(U) = \beta_0 + \beta_1 (U + U^*) + \beta_2 U^2 . \tag{3.1}$$

Although only β_1 and β_2 are dynamically relevant, we keep the normalization β_0 for convenience. The reality condition of eq. (2.3) requires the coefficients of $U^* = U^3$ and U to be equal.

This model has several interesting limits. When β_2 vanishes we have the usual Wilson Z_4 model with a first order transition at the self dual point [10, 11] $\beta_1 = \frac{1}{2} \ln(1 + \sqrt{2})$. For vanishing β_1 we have a Z_2 theory with an irrelevant double covering of the group. In this case there is a single first order transition at [11, 12] $\beta_2 = \frac{1}{2} \ln(1 + \sqrt{2})$. Finally, consider the limit $\beta_2 \to \infty$. In this case every plaquette is driven to lie in $Z_2 = \{\pm 1\}$. Up to a gauge transformation, each link is in Z_2 and we again have the Z_2 transition at $\beta_1 = \frac{1}{4} \ln(1 + \sqrt{2})$.

We now proceed to the kinematics. The parameters b_n are

$$b_0 = \tfrac{1}{4} e^{\beta_0} \left(e^{2\beta_1 + \beta_2} + 2 e^{-2\beta_2} + e^{-2\beta_1 + \beta_2} \right) , \tag{3.2}$$

$$b_1 = \tfrac{1}{4} e^{\beta_0} \left(e^{2\beta_1 + \beta_2} - e^{-2\beta_1 + \beta_2} \right) , \tag{3.3}$$

$$b_2 = \tfrac{1}{4} e^{\beta_0} \left(e^{2\beta_1 + \beta_2} - 2 e^{-\beta_2} + e^{-2\beta_1 + \beta_2} \right) . \tag{3.4}$$

The dual variables are

$$\tilde{b}_0 = \tfrac{1}{2}(b_0 + 2b_1 + b_2) = \tfrac{1}{2}\,e^{\beta_0 + 2\beta_1 + \beta_2}, \tag{3.5}$$

$$\tilde{b}_1 = \tfrac{1}{2}(b_0 - b_2) = \tfrac{1}{2}\,e^{\beta_0 - \beta_2}, \tag{3.6}$$

$$\tilde{b}_2 = \tfrac{1}{2}(b_0 - 2b_1 + b_2) = \tfrac{1}{2}\,e^{\beta_0 - 2\beta_1 + \beta_2}. \tag{3.7}$$

The model is self dual with $b_n = \tilde{b}_n$ whenever

$$b_0 - 2b_1 - b_2 = 0. \tag{3.8}$$

In terms of the β parameters this reduces to the curve

$$\beta_2 = -\tfrac{1}{2}\log\sinh 2\beta_1. \tag{3.9}$$

Inverting equations (3.5)–(3.7) gives

$$\beta_0 = \tfrac{1}{4}\log(8\tilde{b}_0\tilde{b}_1\tilde{b}_2), \tag{3.10}$$

$$\beta_1 = \tfrac{1}{4}\log(\tilde{b}_0/\tilde{b}_2), \tag{3.11}$$

$$\beta_2 = \tfrac{1}{4}\log(\tilde{b}_0\tilde{b}_2/\tilde{b}_1^2). \tag{3.12}$$

The renormalization group functions are now easily found

$$a\frac{\mathrm{d}}{\mathrm{d}a}\beta_0 = 2\beta_0 + \tfrac{1}{4}(\tilde{b}_0^{-1} + 2\tilde{b}_1^{-1} + \tilde{b}_2^{-1})b_0 \ln b_0 + \tfrac{1}{2}(\tilde{b}_0^{-1} + \tilde{b}_1^{-1})b_1 \ln b_1$$
$$+ \tfrac{1}{4}(\tilde{b}_0^{-1} - 2\tilde{b}_1^{-1} + \tilde{b}_2^{-1})b_2 \ln b_2, \tag{3.13}$$

$$a\frac{\mathrm{d}}{\mathrm{d}a}\beta_1 = 2\beta_1 + \tfrac{1}{4}(\tilde{b}_0^{-1} - \tilde{b}_2^{-1})(b_0 \ln b_0 + b_2 \ln b_2) + \tfrac{1}{2}(\tilde{b}_0^{-1} + \tilde{b}_2^{-1})b_1 \ln b_1, \tag{3.14}$$

$$a\frac{\mathrm{d}}{\mathrm{d}a}\beta_2 = 2\beta_2 + \tfrac{1}{4}(\tilde{b}_0^{-1} - 2\tilde{b}_1^{-1} + \tilde{b}_2^{-1})b_0 \ln b_0 + \tfrac{1}{2}(\tilde{b}_0^{-1} - \tilde{b}_2^{-1})b_1 \ln b_1$$
$$+ \tfrac{1}{4}(\tilde{b}_0^{-1} + 2\tilde{b}_1^{-1} + \tilde{b}_2^{-1})b_2 \ln b_2. \tag{3.15}$$

As β_0 is just an overall normalization, we will concentrate on the equations for β_1 and β_2 in sect. 4.

4. Flows and fixed points

Fixed points of the recursion relation occur when the physically relevant couplings β_1 and β_2 are invariant under scale changes. Thus we are interested in simultaneous zeros of the right-hand sides of eqs. (3.14) and (3.15). These occur at:

(A) the self-dual point of the Wilson model,

$$\beta_1 = \tfrac{1}{2}\log(1+\sqrt{2}) = 0.44068\ldots,\ \beta_2 = 0\,;$$

(B) the self-dual point of the Potts action,

$$\beta_1 = \beta_2 = -\tfrac{1}{2}\log\sinh\beta_i = 0.27465\ldots\,;$$

(C) the Ising transition on the β_2 axis,

$$\beta_1 = 0\,, \qquad \beta_2 = \tfrac{1}{2}\log(1+\sqrt{2}) = 0.44068\ldots\,;$$

(D) the Ising transition at

$$\beta_2 = \infty\,, \qquad \beta_1 = \tfrac{1}{4}\log(1+\sqrt{2}) = 0.22034\ldots\,;$$

(E) the strong coupling point, $\beta_1 = \beta_2 = 0$;
(F) $\beta_1 = 0$, $\beta_2 = \infty$;
(G) $\beta_1 = \infty$, $\beta_2 = 0$;
(H) $\beta_1 = \infty$, $\beta_2 = +\infty$;
(I) $\beta_1 = \infty$, $\beta_2 = -\infty$.

If we start at some point in the β_1, β_2 plane and integrate eqs. (3.14) and (3.15), we obtain flow curves which connect various of these fixed points as a runs from 0 to infinity. In fig. 1 we show this plane along with several typical flow curves. Separatrices connect several of these fixed points and divide the coupling space into distinct regions. The solid curves in fig. 1 represent the predicted critical lines. Integrating the renormalization group equations to large lattice spacing yields sharply different final results depending on which side of one of these critical lines one begins.

The renormalization group equations predict the exponents for the divergence of the correlation length near the critical lines. As we approach a critical line along a trajectory which intersects it in a non-tangential way, we characterize the correlation length divergence

$$\xi \propto \left(\beta_i - \beta_{i_c}\right)^{-\nu}. \tag{4.1}$$

A flow curve from near a critical line will pass near one of the fixed points as we integrate out degrees of freedom. The closer one is to the critical line, the more

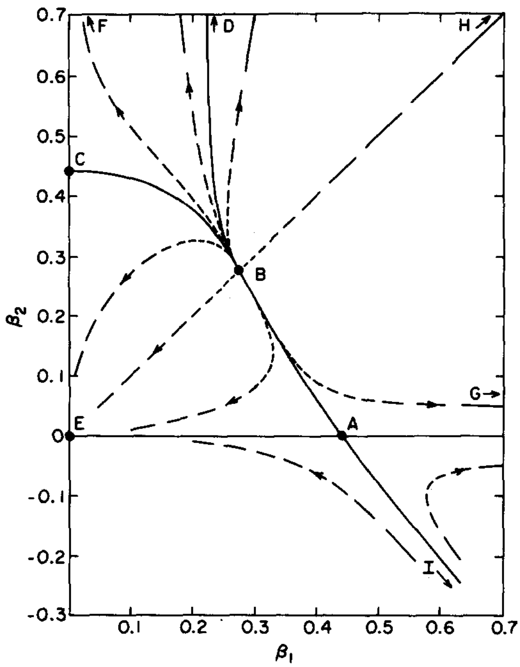

Fig. 1. Renormalization group flows for the Z_4 system. The points A-I are the fixed points discussed in the text. The solid curves are the predicted phase transition lines. The dashed curves are typical flows, directed towards increasing lattice spacing. The length of the dashes is proportional to the flow rate.

"time" in the sense of the variable a is spent flowing in the vicinity of the critical point. Thus the value of ν is determined by the behavior of the flow curves near the fixed point. We are led to linearize the renormalization group equations near a fixed point β_i.

$$a\frac{\mathrm{d}}{\mathrm{d}a}\left(\beta_i-\beta_i^{\mathrm{F}}\right)=M_{ij}\left(\beta_j-\beta_j^{\mathrm{F}}\right)+\mathrm{O}\left(\left(\beta-\beta^{\mathrm{F}}\right)^2\right). \tag{4.2}$$

The matrix M may then be diagonalized, and its positive eigenvalues represent the critical exponents. Thus, near the critical line connecting points A and B, the flow curves pass near point A. The corresponding matrix M has one positive eigenvalue

$$\nu=0.7535\ldots, \tag{4.3}$$

with eigenvector parallel to the β_1 axis. This same eigenvalue determines the critical behavior on the other side of point A, running to point I. In a similar manner we find the exponent along the dual lines BC and BD from the positive eigenvalue at point C. This gives numerically the same exponent as in eq. (4.3).

We now turn to fixed point B, which has two positive eigenvalues. This point is ultraviolet attractive in all directions. The larger of these eigenvalues represents approach along the Potts line $\beta_1 = \beta_2$ and gives a new exponent

$$\nu_P = 0.914. \tag{4.4}$$

Note that in this simple example we see the prediction of universality. For the range of couplings along, say, the line AB, we have a universal critical exponent determined by fixed point A. Thus any one parameter model representing a trajectory crossing this line should be in the same "universality class".

5. Discussion

Recent Monte Carlo studies have investigated the full two coupling Z_4 gauge model [9]. The renormalization group predictions presented here give some but not all features of the phase diagram. The splitting of the self dual transition AB into two mutually dual Z_2 like transitions indeed occurs at a triple point, although this point appears in actuality to lie above the Potts point. In the gauge model all these transitions are first order, rather than the predicted second order. This problem with the Migdal relations for a gauge theory was already observed with the simple Z_2 system.

The most glaring feature missed by the recursion relation is the splitting of the self dual transition at negative β_2 to give an intermediate Coulomb phase. The Monte Carlo results also indicated another first order transition line originating from the Z_2 transition on the negative β_2 axis. This line lies entirely in a region where the parameter b_2 is negative. Here the infinitesmal form of the recursion relation must break down because the logarithms in eq. (2.18) become complex.

In conclusion, we feel that the Migdal-Kadanoff recursion relations provide a simple tool which rather rapidly makes predictions on rather complex phase diagrams. The results can miss certain subtle details, but, in the same spirit as mean field theory, the approximation gives a qualitative understanding of many gross features.

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