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Lattice Gauge Theory – Present Status

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ABSTRACT

Lattice gauge theory is our primary tool for the study of non-perturbative phenomena in hadronic physics. In addition to giving quantitative information on confinement, the approach is yielding first principles calculations of hadronic spectra and matrix elements. After years of confusion, there has been significant recent progress in understanding issues of chiral symmetry on the lattice.

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Lattice gauge theory is a rather old subject, going back to Wilson's work of the early 70's [1]. Through the 80's it grew into a major industry. The field is currently dominated by computer simulations, although it is in fact considerably broader. The main results are now presented annually at a lattice conference attended by about 300 participants. The proceedings of these meetings make a good source of background material for the topic [2].

The lattice program has rather grandiose goals: the first principles solution of hadronic physics. Indeed, sometimes the practitioners get a bit overenthusiastic in stating what is possible. On the other hand, there is no other known way to obtain many of the quantities currently being calculated.

Among these quantities are the hadronic spectra, the hadronic matrix elements of operators of importance to weak decays, and the properties of the quark gluon plasma. Going beyond hadrons, there has been extensive work on obtaining constraints on the Higgs particle. There have also been simulations of curved space times, made with the hope of getting a handle on quantum gravity.

So why do we put our field theory on a lattice? From my point of view, this is a mathematical trick. The lattice provides an ultraviolet cutoff which allows the system to be placed on a computer. Perhaps most important, the cutoff is not based on perturbation theory, and thus we can study non-perturbative physics, such as confinement. Also, unlike with some other non-perturbative schemes, we have a well defined system for study.

Wilson's formulation provides a rather elegant framework for these studies. This begins with a discretization of the action

$$S = \int d^4x F^{\mu\nu} F_{\mu\nu} \rightarrow \frac{\beta}{3} \sum_p \mathrm{ReTr}(U_p) \tag{1}$$

Here the sum is over all elementary squares or "plaquettes" of a four dimensional simple hypercubic lattice. The variable U_p is an $SU(3)$ matrix which measures the gauge field flux through the plaquette, and is the matrix product of elementary link variables surrounding the plaquette

$$U_p = \prod_{\{i,j\} \in p} U_{i,j} \tag{2}$$

where this product is understood in an ordered sense. The individual link variables are identified as the phase factors associated with the gauge field along the respective link

$$U_{i,j} \sim \exp(i \int_{x_i}^{x_j} A^\mu dx_\mu). \tag{3}$$

The bare gauge coupling is related to the parameter β in eq. (1)

$$\beta = \frac{6}{g_0^2} \tag{4}$$

The phenomenon of asymptotic freedom relates the coupling to the lattice spacing a. In particular, the bare coupling should go logarithmically to zero with the scale on which it is defined. Here this scale is the lattice spacing; so, we have

$$\alpha_0 \sim \frac{1}{\beta_0 \ln(1/(a^2\Lambda^2))} \tag{5}$$

where Λ is an integration constant for the renormalization group equation, and β_0 is a numerical constant. Instead of regarding the coupling as a function of the scale, we can invert this relation and consider the lattice spacing as a function of the coupling

$$a \sim \frac{1}{\Lambda} \exp(\frac{-1}{2\beta_0 \alpha_0}). \tag{6}$$

If we now go to our lattice and measure any dimensional quantity in lattice units, these relations give us a handle on the strong coupling constant. Various physical quantities can set the normalization. In the earliest studies of confinement it was usual to take the string tension or Regge slope. People interested in weak interaction matrix elements often pick one of the meson decay constants, such as f_π, to set the scale for other measurements. In a spectroscopic application one might pick a light hadron mass, such as of the rho or nucleon. For another specific example, the Fermilab group recently studied the charmonium spectrum on the lattice and extracted a value for the strong coupling constant [3]. Putting in corrections corresponding to four quark flavors, they quote

$$\alpha_{\overline{MS}}(5\text{Gev}) = 0.174 \pm 0.012 \tag{7}$$

which is in reasonable agreement with experimental measurements.

Although the lattice represents a broad framework for the nonperturbative definition of a field theory, the subject is currently dominated by one approach, that of Monte Carlo simulation. This uses the analogy between the Feynman path integral

$$Z = \int dU e^{\beta/3 \sum \mathrm{ReTr}(U_p)} \tag{8}$$

and the "partition function" for a set of "spins" $\{U\}$ at "temperature" $1/\beta$. This is easily simulated on a finite lattice by standard methods. A Monte Carlo program sweeps over a lattice stored in a computer memory and makes random changes biased by the above "Boltzmann weight." The procedure generates a sequence of configurations which mimic "thermal equilibrium."

$$P(C) \sim e^{\beta \sum_p U_p} \tag{9}$$

One of the captivating features of the technique is that the entire lattice is available in the computer memory; so, in principle one can measure anything. On the other hand, there are inherent statistical fluctuations which may make some things hard to extract. This represents a new aspect of theoretical physics, wherein theorists have statistical errors. In addition these calculations have several sources of systematic errors, such as effects of finite volume and finite lattice spacing. Quark fields introduce further sources of error, including extrapolations from heavy to physical quark masses. Furthermore, many calculations are made feasible by what is termed the valence or quenched approximation, wherein virtual quark loops are neglected.

Indeed, quark fields introduce serious unsolved problems. These problems are by no means new, having been with us since the beginnings of lattice gauge theory. One problem is the issue of finding a reasonable computer algorithm for simulating fermions. In particular, since the quark fields are anticommuting, the full action is not an ordinary number, and the analogy with classical statistical mechanics breaks down. Algorithms in current practice begin by formally integrating out the fermions to give a determinant

$$\begin{aligned} Z &= \int dA\ d\bar\psi d\psi\ \exp(S_g + \bar\psi(\not{D} + m)\psi) \\ &= \int dA\ e^{S_g}\ |\not{D} + m|. \end{aligned} \tag{10}$$

This determinant is, however, of a rather huge matrix, and is quite tedious to simulate. Over the years many clever tricks have been found to simplify the problem, but I regard these approaches as still rather ugly. In addition, if one wants to study physics in a chemical potential, as would represent background baryon density in a heavy ion experiment, then no viable simulation algorithms are known.

The other old problem with quarks has to do with the issue of fermion doubling and and chiral symmetry. Here there has been considerable recent progress, to which I will return later in this talk.

The difficulties with simulating dynamical fermions have led to the majority of simulations being done in the "valence" or "quenched" approximation. Here the feedback of the fermion determinant on the dynamical gauge fields is ignored. Hadrons are studied via quark propagators in a gauge field obtained in a simulation of gluon fields alone. In terms of Feynman diagrams, all gluonic exchanges are included between the constituent quarks, but effects of virtual quark production, beyond simple renormalization of the gauge coupling, are dropped. The primary motivation is the saving of orders of magnitude in computer time. While this may seem a drastic approximation, the fact that the naive quark model works so well hints that things might not be so bad.

One of the longstanding goals of lattice calculations is an understanding of hadronic spectra. If we consider the correlation between some operator ϕ taken at two widely separated points, we expect a generic behavior

$$\langle \phi(x)\phi(0)\rangle \longrightarrow e^{-Mx} \qquad (11)$$

where M is the mass of the lightest hadron which can be created by ϕ acting on the vacuum. Via such calculations using different operators, the masses of a large variety of states can be estimated. In these calculations the bare quark masses are parameters. Lore based on chiral symmetry suggests that as the quark masses go to zero, so will the masses of the corresponding pseudoscalar mesons, i.e. the pions. Thus, the procedure is to adjust the quark masses to get, say, m_π/m_ρ right, and then all other mass ratios, such as m_N/m_ρ should be determined.

An extensive recent calculation of this type was presented in ref. [4]. This particular project was carried out on a specially built computer which ran for one year at an average speed of six gigaflops. These valence approximation results for the light hadron masses are consistent with experiment to within 6%.

Another area of extensive investigation is the phase transition of the vacuum to a plasma of free quarks and gluons at a temperature of a few hundred MeV. To study field theory at a finite temperature, we use the fact that in a finite temporal box of length t the path integral takes the form

$$Z \sim \exp(-Ht) \tag{12}$$

where H is the quantum Hamiltonian. This is exactly the thermal partition function. Both theoretical analysis and numerical simulations have shown the existence of a high temperature regime wherein confinement is lost and chiral symmetry is manifestly restored [5]. The lattice Monte Carlo calculations have given us the best estimate of the relevant transition temperature

$$\begin{aligned} T_c &\sim 235 \text{ MeV},\ 0 \text{ flavors} \\ T_c &\sim 150 \text{ MeV},\ 2 \text{ flavors}. \end{aligned} \tag{13}$$

The relatively large difference between these numbers was somewhat unexpected. Indeed, this is the only place known where the valence approximation seems to have a substantial physical effect.

A large subindustry in the lattice community is the evaluation of hadronic matrix elements of operators relevant to processes such as weak decays. The standard electroweak theory makes precise predictions for the relevant operators leading to weak decays, but to relate these to observed decay rates requires the inclusion of strong interaction initial and final state corrections. These are non-perturbative in nature, and thus fall directly into the lattice gauge theorist's relm. This is a rather large area of research which I cannot cover adequately here. Instead, I defer to the recent review of Bernard and Soni [6]. To quote one recent result, the decay constants for the B and D mesons have been obtained in Ref. [7]. Using f_π to set the overall scale, they find

$$\begin{aligned} f_B &= 187(10) \pm 34 \pm 15 MeV \\ f_{B_s} &= 207(9) \pm 34 \pm 22 MeV \\ f_D &= 208(9) \pm 35 \pm 12 MeV. \end{aligned} \tag{14}$$

I now return to the problems with fermions and discuss some of the issues concerning chiral symmetry and fermion doubling. It was realized quite early that if one naively discretizes the Dirac equation on the lattice, one obtains extra particles. Wilson showed how these could be removed by adding the so called “Wilson term” which formally vanishes in the continuum limit. Unfortunately, this term inherently violates chiral symmetry. As many predictions have been based on this symmetry, the usual hope is that in continuum limit it will return. Meanwhile, however, there is nothing special about massless quarks, and when the cutoff is still in place one must “tune” the bare parameters to make m_π small.

It has recently been suggested that an infinite tower of heavy states may solve this problem. The basic mechanism is to absorb the extra species of the naive formulation into a band of heavy states. One formulation of the idea was presented by Kaplan [8], and some intriguing variations discussed by Frolov and Slavnov [9] and by Neuberger and Narayanan [10].

Let me discuss the problem in somewhat more detail in one space dimension. A naive discretization of the Dirac Hamiltonian is

$$H_0 = K \sum_j i(a_j^\dagger a_{j+1} - b_j^\dagger b_{j+1}) + M \sum_j a_j^\dagger b_j + h.c. \tag{15}$$

where a_j and b_j are fermionic annihilation operators on sites j located along a line. They represent the upper and lower components of a two component spinor. The spectrum of single particle states for this Hamiltonian is easily found in momentum space

$$E^2 = M^2 + 4K^2 \sin^2(q) \tag{16}$$

where q runs from 0 to 2π. Filling the negative energy states to form a Dirac sea, the physical excitations consist of particles as well as antiparticle “holes.” The doubling problem is manifested in the fact that there are low energy excitations for momenta q in the vicinity of π as well as 0. In D spatial dimensions, this doubling increases to a factor of 2^D.

A simple solution to the doubling was presented some time ago by Wilson, who added a term that created a momentum dependent mass

$$H = H_0 - \; rK \sum_j (a_j^\dagger b_{j+1} + b_j^\dagger a_{j+1} + h.c.) \tag{17}$$

where r is called the Wilson parameter. The energy spectrum is now

$$E^2 = 4K^2 \sin^2(q) + (M - 2Kr\cos(q))^2 \tag{18}$$

and we see that the states at q near π have a different energy than those near 0. For the continuum limit, the parameters should be adjusted so that the extra states become infinitely heavy.

This Hamiltonian has a special behavior when $2Kr = M$. In this case one of the fermion species becomes massless. This provides a mechanism for obtaining light quarks and chiral symmetry. Unfortunately, when the gauge interactions are turned on, the parameters renormalize, and tuning becomes necessary to maintain the massless quarks. This is the basis of the conventional approach to chiral symmetry with Wilson fermions; one tunes the "hopping parameter" K until the pion is massless, and calls that the chiral limit.

The above Hamiltonian has some peculiar properties when we take the hopping parameter larger than the critical value. This region has been discussed by Aoki and Gocksch [11] in the context of a spontaneous breaking of parity, although the connection of their work to what I will discuss below is as yet obscure.

Restricting the above Hamiltonian to a finite box with open boundaries, the states become discrete and fall into three classes. First there is a set of states with positive energy which represent the particle band with energies above $|2Kr - M|$. Second, there are the complimentary negative energy states which represent the Dirac sea. Finally, if K exceeds the critical value $M/2r$, there are two isolated levels left near zero energy. These levels are surface modes bound to the boundary of our finite box. They are split from zero energy by tunnelling from one boundary to the other; indeed, they go to exactly zero energy in the infinite volume limit.

Kaplan [8] has proposed to use such zero modes on a boundary as the basis for a formulation of chiral fermions. The idea is to consider the above one dimensional system as representing not a physical coordinate, but a hypothetical "fifth" dimension. Our world, then, would be a four dimensional interface representing a boundary in this extra dimension. The physical quarks and leptons are the above surface modes. Momentum in the physical dimensions gives an additional contribution to their energy, yielding a conventional relativistic spectrum.

The existence of the surface modes requires that the hopping parameter exceeds a critical value. A more general result is that these modes exist whenever the hopping parameter passes through the critical value as one passes through the surface. I have simplified the discussion by having K vanish outside the supercritical region.

With the physical transverse dimensions included, the critical value for the hopping parameter in the fifth dimension will depend on the physical momenta. By appropriately choosing the parameters, only those states with low physical momentum will satisfy the conditions for a surface mode to exist, and all extra doubler modes can be eliminated [12]. Thus the tuning problem of the conventional approach is replaced with a large volume limit for the new fifth dimension.

In this picture, the opposite walls have modes representing opposite helicity. Chiral anomalies appear through tunnelling between these walls. As discussed in Ref. [13], these anomalies maintain their correct values even as the band of states in the extra dimension goes to infinite mass.

The above discussion ignored the contributions of gauge fields. To avoid adding new unwanted degrees of freedom, it is most natural to place gauge fields only on the links in the physical four dimensions of ordinary space time. It is perhaps simplest to follow Ref. [10] and to think of the extra dimension as representing a "flavor" space, with all the new flavors coupling equivalently to the gauge fields.

One consequence of this picture is that the same gauge field will couple to the surface modes on both walls of a finite slab. Thus in the simplest case both chiralities of the fermions will be present, and one does not have an immediate lattice formulation of the standard model of electroweak interactions. At first sight this appears discouraging; indeed, we know the weak interactions require parity violation, and for conceptual reasons we would like to have a non-perturbative formulation.

Formulating the standard model on the lattice presumably requires some rather subtle features. For example, some time ago t'Hooft [14] showed how instanton effects give rise to a small baryon number violation in the standard model. Any non-perturbative formulation must include such a phenomenon. What is exciting is that this extra dimension formulation may have a mechanism to do just that.

In particular, the t'Hooft baryon violation is an effect of anomalies, and anomalies in this surface picture are represented by tunnelling through the extra dimension. For this to work out, baryon states on on one surface will need to mix with lepton states on the other. Then baryon violation could then arise as a tunnelling between these states through the "fifth" dimension. The details of such a scheme still need to be worked out; indeed, one must carefully assure that no anomalies remain for the gauged currents.

To conclude this talk, let me make a few disconnected remarks. As the lattice provides a non-perturbative definition of a field theory, there have been numerous efforts at using the methods on other models. A particularly active area has been towards understanding gravity. The general idea would be to discretize the points of space-time and then do a sum over curvatures. So far the results of this program have been limited, but no other approach to quantum gravity has yet been successful and the potential payoff is great. A review of the subject was given by Jurkiewicz [15].

Occasionally one hears suggestions that there might indeed be some fundamental lattice at a scale below current observations. My qualms are that this opens up a vast number of variations. In the past the criterion of renormalizability has proven quite useful in limiting the theories used in particle physics. We have no strong evidence, apart from possibly gravity, that nature uses any other fundamental interactions. With a fundamental lattice this need for renormalizability becomes obscured.

Recently there has been considerable interest in fermionic theories based on fundamental four fermion couplings. See for example Ref. [16]. While non-renormalizable, these can give rise to interesting field theoretic phenomena through dynamical symmetry breaking. It is conceivable that lattice methods may be of use here. In such theories, a natural cutoff scale would be the Planck length. Ref. [17] has suggested that such a theory might circumvent the fermion doubling problems of conventional approach.

Finally, let me note that after twenty years, lattice gauge theory remains a thriving industry. The method is still the most viable approach to study non-perturbative phenomena in quantum field theory. Despite these successes, the fundamental problems with lattice fermions show that we still have an acute need for new ideas.

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