

Hidden symmetries in two dimensional field theory

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Abstract

The bosonization process elegantly shows the equivalence of massless scalar and fermion fields in two space-time dimensions. However, with multiple fermions the technique often obscures global symmetries. Witten's non-Abelian bosonization makes these symmetries explicit, but at the expense of a somewhat complicated bosonic action. Frenkel and Kac have presented an intricate mathematical formalism relating the various approaches. Here I reduce these arguments to the simplest case of a single massless scalar field. In particular, using only elementary quantum field theory concepts, I expose a hidden $SU(2) \times SU(2)$ chiral symmetry in this trivial theory. I then discuss in what sense this field should be interpreted as a Goldstone boson.

Key words: bosonization, chiral symmetry, Goldstone bosons

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1 Introduction

A large variety of two dimensional models can be related and often solved via the process of bosonization. [1,2,3,4,5] This process, however, often obscures certain symmetries. For example, the two flavor generalization [6,7] of the Schwinger model [8] in the fermion formulation has an $SU(2) \times SU(2)$ chiral symmetry, but the bosonic solution has one massive and one massless scalar field, both free. In the strong coupling limit the massive particle should become irrelevant, leaving the puzzle of how can a chiral symmetry appear in the trivial theory of only a single free massless scalar.

This question has been discussed in terms of another formulation of bosonization, where the basic fields are elements of a group, and the chiral symmetries

involve rotations of these elements. This “non-Abelian” bosonization [9] involves a chiral Lagrangian containing a rather interesting topological term. [10,11] While this formulation keeps the chiral symmetries more transparent, the mapping between the chiral fields and the alternative Abelian bosonization is somewhat obscure. Some of the connections were discussed in a series of papers by Affleck, [12,13] and an explicit construction of the connection is given by Amaral and StephanyRuiz.[14] Halpern [7] gives the form for the chiral currents in the multi-flavor Schwinger model.

In this paper I return to this old topic with further discussion of how a non-Abelian current algebra is hidden in the simplest two dimensional scalar field theory. The construction is well known to the string theory community and is a special case of the general technique of Frenkel and Kac. [15,16,17] That discussion, however, is placed in a rather formal context; my goal here is to elucidate the surprising consequences more transparently, specializing to the simplest case and using only concepts from elementary quantum field theory.

After a discussion of the connection with Witten’s non-Abelian formulation, I turn to some comments on the role and counting of Goldstone bosons. In two dimensions the definition of a Goldstone boson is subject to some interpretation. On the one hand, infrared fluctuations preclude the matrix valued fields from acquiring a vacuum expectation value. This is the basis of the familiar arguments that spontaneous breaking of a continuous symmetry cannot occur in two dimensions.[18] On the other hand, the chiral charges are rather singular objects, and with a simple cutoff the vacuum is not annihilated by them, even in the limit that the cutoff is removed. The latter is sufficient to require the existence of a massless particle in the spectrum, and forms the basis of one proof of the Goldstone theorem.[19] In this sense a two dimensional field theory can exhibit a Goldstone boson, although it must be free.

Motivated by the simplest case of the strongly coupled two flavor Schwinger model, I will show that the trivial field theory of a free massless boson in one space dimension has a hidden $SU(2) \times SU(2)$ symmetry. In particular, I will construct conserved currents $J_{R,\mu}^\alpha(x)$ and $J_{L,\mu}^\alpha(x)$, where α is an “isospin” index running from 1 to 3, $\mu \in \{0, 1\}$ is a Lorentz index, and L, R label left and right handed parts. The resulting charge densities satisfy the equal time commutation relations

$$\begin{aligned}
[J_{R,0}^\alpha(x), J_{R,0}^\beta(y)] &= i\epsilon^{\alpha\beta\gamma} J_{R,0}^\gamma(x) \delta(x-y) + A\delta^{\alpha\beta} \partial_x \delta(x-y), \\
[J_{L,0}^\alpha(x), J_{L,0}^\beta(y)] &= i\epsilon^{\alpha\beta\gamma} J_{L,0}^\gamma(x) \delta(x-y) - A\delta^{\alpha\beta} \partial_x \delta(x-y), \\
[J_{R,0}^\alpha(x), J_{L,0}^\beta(y)] &= 0.
\end{aligned}
\tag{1}$$

The notation left or right indicates that the corresponding currents are to be constructed from operators involving particles moving to the right or left,

respectively. I include here a Schwinger [20] term with coefficient A that will be determined. The Schwinger terms for the two chiralities differ in sign, assuring that they cancel in the commutators of the vector current.

I organize this paper as follows. In Section 2 I make a few remarks on why these two dimensional models may be useful for the understanding of chiral symmetry in four dimensions. Section 3 I establish notation by defining the Hilbert space in which I work. Section 4 discusses splitting the massless fields into left and right moving parts. With Section 5 I review the concept of normal-ordered exponentiated fields. The basic result for the currents appears in Section 6. I then turn to the connection with group valued fields and the corresponding equations of motion in Section 7. Section 8 explores the question of whether the underlying massless field can be thought of as a Goldstone boson. A few final comments appear in Section 9.

2 Four dimensional motivations

Of course we live in a four dimensional world; so, it is perhaps worth mentioning some of the reasons these two dimensional models are worth studying. Two dimensional field theories are of intense interest to the string theory community, where the essence of these results is well known. However, my interest comes from a rather different direction, related to attempts to formulate quantum field theory beyond the realm of perturbation theory.

For the strong interactions, spontaneous breaking of a global chiral symmetry plays a major role in the understanding of the hadronic spectrum. Indeed, the lightness of the pions relative to the rho mesons has long been explained in this framework. However, the phenomenon is inherently not perturbative. The lattice provides the primary non-perturbative method in field theory, but issues related to anomaly cancellation make that approach rather complicated. [21] Effective chiral Lagrangians provide another powerful route to non-perturbative information, although in a less quantitative manner due to increasing numbers of arbitrary parameters at higher order. To understand these issues better, the solvable models in two dimensions can provide insight into how chiral symmetry works. The symmetries discussed here are also in the four dimensional quark-gluon theory with two massless flavors, although the counting of Goldstone bosons manifests itself somewhat differently. While much of the rigorous mathematical work on these two dimensional models is built on conformal symmetry, this is of less relevance in four dimensions. Indeed, one of the remarkable properties of quark confining dynamics is how asymptotic freedom manages to avoid the conformal symmetry of the classical theory with massless quarks.

Understanding chiral symmetry is presumably an important step towards a non-perturbative formulation of chiral gauge theories. As the weak interactions do not conserve parity, we know that the gauge fields are coupled in a chirally non-symmetric way to the fundamental fermions. Without a non-perturbative formulation, one might worry whether the standard model is well defined as a field theory. But here the lattice issues are unresolved; a fully finite lattice regularization that preserves an exact underlying gauge symmetry remains elusive. This is in contrast to the bosonization technique, around which this discussion revolves, where there are no problems defining generalizations of the Schwinger model to a chiral theory, as long as anomalies are properly cancelled.[22] Indeed, this shows that the absence of a clean lattice regulator does not preclude the existence of at least these simplified chiral theories.

3 The scalar field

To establish notation, in this section I set up the basic scalar field theory in terms of which I will construct the non-Abelian currents. I work at a fixed time in a Hilbert space formulation. The states of this space are generated by bosonic creation operators a_p^\dagger operating on a normalized vacuum state $|0\rangle$. The momentum space commutation relations are the usual

$$[a_p, a_{p'}^\dagger] = 4\pi p_0 \delta(p - p'), \quad (2)$$

and the vacuum is annihilated by the destruction operators, $a_p|0\rangle = 0$. As I have in mind the massless theory, I take $p_0 = |p|$. The local field and its conjugate momentum are

$$\begin{aligned} \Phi(x) &= \int_{-\infty}^{\infty} \frac{dp}{4\pi p_0} \left(e^{-ipx} a_p + e^{ipx} a_p^\dagger \right), \\ \Pi(x) &= i \int_{-\infty}^{\infty} \frac{dp}{4\pi} \left(e^{-ipx} a_p - e^{ipx} a_p^\dagger \right). \end{aligned} \quad (3)$$

These satisfy the canonical position space commutation relations

$$[\Pi(x), \Phi(y)] = i\delta(x - y).$$

For a massless particle, time evolution is given by the simple free field Hamiltonian

$$H = \int dx \left(: \Pi^2(x) : + : (\partial_x \Phi(x))^2 : \right) = \int \frac{dp}{4\pi} a_p^\dagger a_p. \quad (4)$$

The colons denote normal ordering with respect to the creation and annihilation operators, i.e. all annihilation operators are placed to the right of all creation operators. This normal ordering ensures a zero energy vacuum.

As is well known,[18] a massless field in two dimensions is a rather singular object. In particular, the two point function $\langle 0|\Phi(x)\Phi(y)|0\rangle$ has an infrared divergence. This can be circumvented by considering correlations between derivatives of the field, which are better behaved. I will shortly introduce infrared and ultraviolet cutoffs giving well defined field correlators. Any final conclusions require combinations of the fields having a finite limit as these cutoff parameters are removed.

4 Chiral fields

In one dimension there is a natural notion of chirality for massless particles. A particle going to the right in one frame does so at the speed of light in all frames. This is true regardless of whether the particle is a boson or a fermion. Thus it is natural to separate the field into right and left moving parts

$$\Phi(x) = \Phi_R(x) + \Phi_L(x), \tag{5}$$

where the right handed piece only involves operators for positive momentum

$$\Phi_R(x) = \int_0^\infty \frac{dp}{4\pi p_0} \left(e^{-ipx} a_p + e^{ipx} a_p^\dagger \right). \tag{6}$$

Correspondingly, the left handed field only involves negative momentum. The goal here is to construct the right (left) handed currents using only the right (left) handed field.

Note that the canonical momentum satisfies

$$\Pi(x) = -\partial_x(\Phi_R(x) - \Phi_L(x)). \tag{7}$$

Thus one can alternatively work with $\{\Pi(x), \Phi(x)\}$ or $\{\Phi_R(x), \Phi_L(x)\}$ as a complete set of operators in the Hilbert space. Also note that formally $\Phi_R(x)$ does not commute with itself at different positions. However derivatives of the field do, and, as mentioned above, only derivatives of the field are physically sensible. With this proviso, either the left or right fields define a relativistic quantum field theory on their own. Were a mass present, the left and right fields would mix under Lorentz transformations and should not be considered independently.

Under the Hamiltonian of Eq. (4), the equations of motion for the chiral fields are particularly simple. The right (left) field only creates right (left) moving waves, or in equations

$$\begin{aligned}(\partial_t + \partial_x)\Phi_R(x) &= 0, \\(\partial_t - \partial_x)\Phi_L(x) &= 0.\end{aligned}\tag{8}$$

For the time being I will concentrate on the right handed field.

As with the full field, correlation functions of these fields are divergent. To get things under better control, I introduce an infrared cutoff m and an ultraviolet cutoff ϵ with the definition

$$\Phi_R(x) = \int_m^\infty \frac{dp}{4\pi p} e^{-\epsilon p/2} \left(e^{-ipx} a_p + e^{ipx} a_p^\dagger \right).\tag{9}$$

Both cutoffs are to be taken to zero at the end of any calculation of physical relevance.

There is some arbitrariness in both these cutoffs. In particular, another popular infrared cutoff gives the scalar boson a small physical mass via the choice $p_0 = \sqrt{p^2 + m^2}$. All the following discussion could be done either way. A physical mass, however, complicates the separation of chiral parts, since Lorentz transformations will mix them. With the choice taken here, the left and right movers remain independent, although Lorentz transformations will change the cutoff.

With the cutoffs in place, the correlation of two of these operators becomes well defined

$$\begin{aligned}\Delta_R(x-y) &= \langle 0 | \Phi_R(x) \Phi_R(y) | 0 \rangle = \int_m^\infty \frac{dp}{4\pi p} e^{-ip(x-y)} \\ &= \frac{1}{4\pi} \left(C - \log(x-y-i\epsilon) - \log(m) - \frac{i\pi}{2} \right).\end{aligned}\tag{10}$$

Here C is the Gompertz constant[23] divided by e and has the value

$$C = \int_1^\infty \frac{dp}{p} e^{-p} = 0.21938\dots\tag{11}$$

Note that this “propagator” diverges logarithmically as m goes to zero, although its derivatives do not. For example,

$$\langle 0 | \partial_x \Phi_R(x) \partial_y \Phi_R(y) | 0 \rangle = \frac{-1}{4\pi(x-y-i\epsilon)^2} \quad (12)$$

remains a tempered distribution as ϵ goes to zero.

5 Exponentiated fields

The usual construction of fermionic operators in the bosonization process involves exponentiated scalar fields. This will also be the case for the currents below, although detailed factors will differ. To keep things well defined, I consider the normal-ordered operator with the cutoffs in place

$$: e^{i\beta\Phi_R(x)} : = \exp\left(i\beta \int_m^\infty \frac{dp e^{-\epsilon p/2}}{4\pi p} e^{ipx} a_p^\dagger\right) \exp\left(i\beta \int_m^\infty \frac{dp e^{-\epsilon p/2}}{4\pi p} e^{-ipx} a_p\right). \quad (13)$$

The expression in Eq. (10) for the propagator gives a rather simple relation to normal order the product of two of these operators

$$\begin{aligned} : e^{i\beta\Phi_R(x)} : : e^{i\beta'\Phi_R(y)} : &= : e^{i\beta\Phi_R(x)+i\beta'\Phi_R(y)} : \exp(-\beta\beta' \Delta_R(x-y)) \\ &=: e^{i\beta\Phi_R(x)} e^{i\beta'\Phi_R(y)} : \left(\frac{-ie^C}{m(x-y-i\epsilon)}\right)^{-\beta\beta'/4\pi}. \end{aligned} \quad (14)$$

I will always be working with $\beta\beta'$ an integer multiple of 4π ; thus, there is no phase ambiguity. This will be the key relation in the following.

In two dimensions the free massless fermion propagator is proportional to $1/(x-y)$. This is the basis of the usual bosonization which takes $\beta = 2\sqrt{\pi}$ and identifies

$$\psi_R(x) = \frac{e^{C/2}}{\sqrt{2\pi}} \lim_{m \rightarrow 0} \sqrt{m} : e^{2i\sqrt{\pi}\Phi_R(x)} : . \quad (15)$$

The factor in front gives conventionally normalized fermionic commutation relations, and an additional phase appears between the left and right handed fermion fields to have them anticommute. However my goal here is not the fermion field, but rather the non-Abelian currents.

Note that if $-\beta\beta' = 8\pi$ the spatial dependence in Eq. (14) is proportional to that of the correlator between two derivatives of the field as given in Eq. (12). This lies at the heart of the construction of the currents in the next section.

6 The currents

I concentrate here on constructing the right handed current from the right handed field. The construction of the left handed current proceeds in a parallel fashion. I start by selecting one component of the isovector current as the trivially conserved

$$J_{R,\mu}^3 = -k\epsilon_{\mu\nu}\partial_\nu\Phi_R(x). \quad (16)$$

Here $\epsilon_{\mu\nu}$ is the antisymmetric tensor with $\epsilon_{0,1} = 1$ and k is a normalization factor that will shortly be determined. Note that because this involves derivatives, correlation functions of this current are well defined tempered distributions. My convention on repeated Lorentz indices is understood as a summation with the metric $g_{00} = -g_{11} = 1$. Thus the charge density is $J_{R,0}^3(x) = k\partial_x\Phi_R(x)$, and the associated charge is

$$Q^3 = \int_{-\infty}^{\infty} dx J_{R,0}^3(x) = k(\Phi_R(\infty) - \Phi_R(-\infty)). \quad (17)$$

The commutator of the current with itself gives the coefficient of the Schwinger term

$$\begin{aligned} [J_{R,0}^3(x), J_{R,0}^3(y)] &= k^2\partial_x\partial_y(\Delta(x-y) - \Delta(y-x)) \\ &= \frac{k^2}{4\pi}\partial_x\left(\frac{1}{x-y-i\epsilon} - \frac{1}{x-y+i\epsilon}\right) \rightarrow \frac{k^2}{2}\partial_x\delta(x-y). \end{aligned} \quad (18)$$

This relates the coefficient in Eq. (1) to the normalization k (still to be determined), $A = \frac{k^2}{2}$.

For the other components of the currents it is easiest to work with raising and lowering combinations $J_{R,\mu}^\pm = \frac{1}{\sqrt{2}}(J_{R,\mu}^1 \pm iJ_{R,\mu}^2)$. Then the desired commutation relations reduce to

$$\begin{aligned} [J_{R,0}^3(x), J_{R,0}^\pm(y)] &= \pm J_{R,0}^\pm(x) \delta(x-y), \\ [J_{R,0}^+(x), J_{R,0}^-(y)] &= J_{R,0}^3(x) \delta(x-y) + A\partial_x\delta(x-y). \end{aligned} \quad (19)$$

The first relation in conjunction with Eq. (17) indicates that J^+ must induce a “kink” of size $1/k$ in the field Φ_R . The fermionic operators in Abelian bosonization do something similar, but the size of the kink differs. This observation suggests I try the form

$$J_{R,0}^+(x) \sim : e^{i\beta\Phi_R(x)} : . \quad (20)$$

Since the currents should commute at non-vanishing separation, I should take $\beta^2 = 8\pi n$ with n an integer. For the remainder of this discussion I take the lowest value

$$\beta = 2\sqrt{2\pi}. \quad (21)$$

This is the square root of two times the value taken to construct fermion fields. This follows intuitively from the fact that the current should be a fermion bilinear with two “kinks” in orthogonal directions.

For the first commutator in Eq. (19) look at

$$\begin{aligned} [\partial_x \Phi_R(x), : e^{i\beta\Phi_R(y)} :] &= i\beta : e^{i\beta\Phi_R(y)} : \partial_x (\Delta(x-y) - \Delta(y-x)) \\ &= \sqrt{2\pi} : e^{i\beta\Phi_R(y)} : \delta(x-y). \end{aligned} \quad (22)$$

This says I should take $k = 1/\sqrt{2\pi}$ and

$$J_{R,0}^3(x) = \frac{1}{\sqrt{2\pi}} \partial_x \Phi_R(x). \quad (23)$$

To get the normalization of the other currents, use Eq. (14) to work out the commutator

$$\begin{aligned} [: e^{i\beta\Phi_R(x)} :, : e^{-i\beta\Phi_R(y)} :] &= \\ : e^{i\beta(\Phi_R(x) - \Phi_R(y))} : &= \frac{-e^{2C}}{m^2} \left(\frac{1}{(x-y-i\epsilon)^2} - \frac{1}{(x-y+i\epsilon)^2} \right). \end{aligned} \quad (24)$$

As ϵ becomes small the last factor becomes a derivative of a delta function

$$\frac{1}{(x-y-i\epsilon)^2} - \frac{1}{(x-y+i\epsilon)^2} \rightarrow -2\pi i \frac{d}{dx} \delta(x-y). \quad (25)$$

Using the Leibnitz rule

$$f(x) \delta'(x) = -f'(0) \delta(x) + f(0) \delta'(x), \quad (26)$$

the commutator reduces to

$$[: e^{i\beta\Phi_R(x)} :, : e^{-i\beta\Phi_R(y)} :] = \frac{2\pi e^{2C}}{m^2} \left(4\pi\delta(x-y) J_{R,0}^3(x) + i\delta'(x-y) \right). \quad (27)$$

Absorbing the prefactors, the desired commutation follows with

$$J_{R,0}^+(x) = \frac{e^{-C}}{2\pi\sqrt{2}} \lim_{m \rightarrow 0} m : e^{i\beta\Phi_R(x)} :. \quad (28)$$

With Eq. (23), this is the final result. The Schwinger term appears with the same coefficient as in Eq. (18), serving as a consistency check on the normalizations. Finally, note that the equations of motion give the spatial component of the currents $J_{R,1}^\alpha = J_{R,0}^\alpha$ (For the left currents there is a relative minus sign in this relation).

7 Matrix fields

The non-Abelian bosonization of Witten is formulated in terms of a matrix valued field in the fundamental representation of the symmetry group. As I now have the basic symmetry generators, I should be able to construct this matrix field from the elementary scalar as well. This section outlines the procedure. The construction gives rise to an equation of motion that contains a Wess-Zumino term.

Motivated by Witten's[9] discussion, I start by looking for a matrix valued field as a product

$$g = g_L g_R, \quad (29)$$

where g_R (g_L) is constructed from right (left) fields alone. I ask that these satisfy the equations

$$\begin{aligned} \partial_x g_R &= g_R \sigma^\alpha J_{R,0}^\alpha \\ \partial_x g_L &= \sigma^\alpha J_{L,0}^\alpha g_L. \end{aligned} \quad (30)$$

These are solved as “ X ordered” integrals,[24] *i.e.*

$$g_R(x) = X \left(\exp \left(i \int_{-\infty}^x dx' \sigma^\alpha J_R^\alpha(x') \right) \right). \quad (31)$$

The left field would be formulated in the corresponding way with an an “anti- X ordered” integral. This is in direct analogy with path ordered products of group elements in gauge theory. I assume here that the X ordered integration starts with the unit matrix at $-\infty$. Later I will argue that the details of the boundary condition are unimportant since the correlations of these matrix fields decrease to zero with separation.

Being constructed only from right moving fields, g_R should satisfy the relation $(\partial_t + \partial_x)g_R = 0$. Similarly the left field satisfies $(\partial_t - \partial_x)g_L = 0$. For the product we thus wind up with the equation of motion

$$\partial_\mu \partial_\mu g = ((\partial_t + \partial_x)g)g^\dagger((\partial_t - \partial_x)g) = (\partial_\mu g)g^\dagger \partial_\mu g + \epsilon_{\mu\nu}(\partial_\mu g)g^\dagger \partial_\nu g. \quad (32)$$

The piece involving $\epsilon_{\mu\nu}$ is the Wess-Zumino term. Witten[9] has extensively discussed how to obtain these equations of motion from a Lagrangian including a topological term.

Note that given the field g , it is straightforward to go back and reconstruct the currents

$$\begin{aligned} j_{R,\mu}^\alpha &= \frac{-1}{4} \text{Tr} \sigma^\alpha g^\dagger (\epsilon_{\mu\nu} \partial_\nu + \partial_\mu) g \\ j_{L,\mu}^\alpha &= \frac{-1}{4} \text{Tr} g^\dagger \sigma^\alpha (\epsilon_{\mu\nu} \partial_\nu - \partial_\mu) g. \end{aligned} \quad (33)$$

8 Goldstone bosons

My starting point was a massless field. Is it a Goldstone boson? This question is subtle due to the borderline nature of two dimensions. Mermin and Wagner[25] showed that in one space dimension one cannot have ferromagnetism in the sense of an order parameter with a continuous symmetry acquiring an expectation value. Coleman[18] proved this result in the framework of relativistic quantum field theory, relating it to the singular nature of the propagator for a massless field. Affleck[13] rephrases the Mermin Wagner argument in terms of the correlator of two matrix valued fields decreasing to zero at long distances.

On the other hand, the basic proof of the Goldstone theorem[19] considers a symmetry generator that does not annihilate the vacuum. If you have a such a charge that commutes with the Hamiltonian, there must exist states of arbitrarily low energy. These can be created by applying the local charge density times a slowly varying test function on the vacuum. In a theory of particles, states of arbitrarily low energy are built from massless particles moving with arbitrarily low momentum. These are the Goldstone bosons.

So, in this massless theory, do the relevant charges annihilate the vacuum? Generally global charges are somewhat singular objects due to infrared issues.[26] Local charge densities, being derivatives of the field, are well defined. Working directly with them I consider damping the third charge with a Gaussian factor

$$Q_R^3 = \int dx e^{-\alpha x^2} J_{R,0}^3(x) = \frac{-i}{\sqrt{2\alpha}} \int_0^\infty \frac{dp}{4\pi} e^{-p^2/4\alpha} (a_p - a_p^\dagger) \quad (34)$$

As this is linear in the creation/annihilation operators, its vacuum expectation value vanishes. However, its application on the vacuum does not itself vanish, as can be seen from evaluating

$$\langle 0|Q^3 Q^3|0\rangle = \frac{1}{8\pi\alpha} \int_0^\infty p dp e^{-p^2/2\alpha} = \frac{1}{8\pi}. \quad (35)$$

The damping factor cancels out, giving a finite non-vanishing result as it is removed. To the extent that this limit defines the charge, it does not annihilate the vacuum. This is enough to show that states of arbitrarily low energy must exist. In particular, the state $Q_3|0\rangle$, with the cutoff α above in place, is orthogonal to the vacuum but has an expectation value for the Hamiltonian that goes to zero as α does, i.e.

$$\langle 0|Q^3 H Q^3|0\rangle = \frac{1}{8\pi\alpha} \int_0^\infty p^2 dp e^{-p^2/2\alpha} = \frac{\sqrt{\alpha}}{8\sqrt{2\pi}} \rightarrow 0. \quad (36)$$

These states are long wavelength modes of the starting massless field, and by this interpretation it is indeed a Goldstone boson.

The full current algebra consists of three isospin currents. However, this borderline case of two dimensions allows a single boson to suffice for all. This contrasts with higher dimensions where spontaneous symmetry breaking is better defined and there are three independent Goldstone bosons for the $SU(2)$ case.

The Merman-Wagner and Coleman discussions point out that, unlike in higher dimensions, this breaking of symmetry does not appear in the expectation value of the group valued field g . In more dimensions g is usually written as the exponentiated Goldstone boson field. But in the two dimensional case the scalar field has infinite fluctuations, as seen in Eq. (10). These infinite fluctuations make the expectation value of g vanish.

9 Final comments

Note that this discussion pays little attention to boundary conditions. Indeed, the fact that the expectation value of g vanishes suggests that they are irrelevant. This is in some contrast to the usual discussion of the Wess-Zumino term where working with a compact space is a basic starting point.

Turning on a small common mass for the fermions in the two flavor Schwinger model will drive g to have an expectation value. Then the low energy spectrum of the theory will indeed have three degenerate light mesons. One is from the fundamental field Φ and represents the neutral pion in analogy to the four dimensional theory. The other two light excitations are solitons representing the charged pions[27].

Given the singular nature of a massless field and the ability to map between rather different looking formulations, one might ask if the number of massless particles is a well defined concept in two dimensions. A simple physical argument that precisely counts the bosons in a theory is to calculate the vacuum energy per unit volume at finite temperature. This gives a Stefan-Boltzmann law, which in two dimensions reads

$$\langle E/V \rangle = \frac{n_b \pi T^2}{6} \tag{37}$$

where n_b is the number of massless particles. Thus the starting theory has indeed only one Goldstone boson.

Note that this finite temperature energy density can be calculated either in terms of the boson field or in terms of the equivalent fermion field. In the latter case the Fermi-Dirac statistics gives a factor of two reduction, but that is cancelled by the presence of both particles and antiparticles in the fermionic formulation.

One motivation for this study was to understand how the $SU(2)$ symmetry of the two flavor Schwinger model appears in the massless boson field of the solution. But a single massless boson is also equivalent to a single free massless Dirac field. So the latter formulation must also have a hidden $SU(2)$ symmetry. This is most easily understood as a symmetry between particles and antiparticles. For every particle state of momentum p there is also the possibility of having an antiparticle of the same momentum. The hidden symmetry puts these two states into a doublet. By combining fermions with antifermions, the $SU(2)$ symmetry does not commute with fermion number. A single free Dirac fermion is equivalent to an isodoublet of Majorana fermions.

The three flavor Schwinger model is solved via its equivalence to one massive

and two massless bosons. Thus the free theory of two massless scalar bosons in two space-time dimensions must have a hidden $SU(3)$ symmetry. Actually the above discussion in terms of fermions and antifermions suggests that this is in fact a subgroup of an even larger hidden $SU(4)$ symmetry.

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