One flavor QCD

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Abstract

One flavor QCD is a rather intriguing variation on the underlying theory of hadrons. In this case quantum anomalies remove all chiral symmetries. This paper discusses the qualitative behavior of this theory as a function of its basic parameters, exploring the non-trivial phase structure expected as these parameters are varied. Comments are made on the expected changes to this structure if the gauge group is made larger and the fermions are put into higher representations.

I. INTRODUCTION

QCD, the non-Abelian gauge theory of quarks and gluons, is now generally regarded as the underlying dynamics of strongly interacting hadrons. It is a very economical theory, with the only parameters being the overall scale, usually called Λ_{qcd}, and the quark masses. Indeed, in the zero quark mass limit we have a theory with essentially no parameters, in that all dimensionless ratios should in principle be determined. With several massless quarks this theory has massless Goldstone bosons representing a spontaneous breaking of chiral symmetry. The fact that the pions are considerably lighter than other hadrons is generally regarded as a consequence of the quark masses being rather small.

The one flavor situation, while not phenomenologically particularly relevant, is fascinating in its own right. In this case quantum mechanical anomalies remove all chiral symmetries from the problem. No massless Goldstone bosons are expected, and there is nothing to protect the quark mass from additive renormalization. It appears possible to have a massless particle by tuning the quark mass appropriately, but this tuning is not protected by any symmetry and occurs at a mass value shifted from zero. The amount of this shift is non-perturbative and scheme dependent.

In the one flavor theory, the classical formulation does have a chiral symmetry in parameter space. If the mass is complexified in the sense described in the next section, physics is naively independent of the phase of this parameter. However, when quantum effects are taken into account, a non-trivial dependence on this phase survives and a rather interesting phase diagram in complex mass appears. A large negative mass should be accompanied by a spontaneous breakdown of parity and charge conjugation symmetry, as sketched in Fig. (1).

Despite the simplicity of this diagram, certain aspects of this theory remain controversial. Does chiral symmetry have anything to say about the massless theory? What does one really mean by a massless quark when it is confined? Is there any sense that a quark condensate can be defined? Is it in any sense an "order parameter"? The purpose of this paper is to provide a framework for discussing these issues by bringing together a variety of arguments that support the basic structure indicated in Fig. (1).

After a brief discussion of the parameters of the theory in Section II, Section III shows how simple effective Lagrangian arguments give the basic structure. Much of that section is adapted from Ref. [1]. A discussion of the Dirac eigenvalue structure for this theory appears in sections IV and V. Small complex eigenvalues are treated separately from the exact zero modes arising from
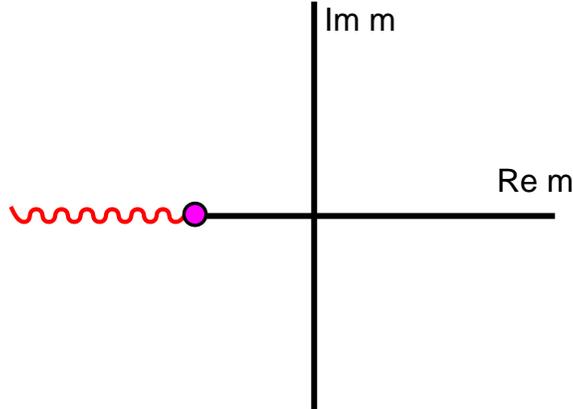
FIG. 1: The phase diagram for one flavor quark-gluon dynamics in the complex mass plane. The wavy line in the negative mass region corresponds to a first order phase transition ending at a second order critical point. The transition represents a spontaneous breaking of CP symmetry. A finite gap separates the transition from vanishing quark mass.

topologically non-trivial gauge fields. These sections expand on the ideas in Ref. [2]. Section VI expands on Ref. [3], reviewing the renormalization group arguments for regularizing the theory, showing how the renormalized mass is defined, and exposing the ambiguity in this definition for the one flavor theory. Section VII addresses expected changes in the basic picture when the size of the gauge group is increased and the fermions placed in higher representations than the fundamental. Brief conclusions are in Section VIII.

II. PARAMETERS

The theory under consideration is motivated by the classical Lagrangian density for $SU(3)$ gauge fields coupled to one species of quark in the fundamental representation of the gauge group

$$L = \frac{1}{4g^2} F_{\mu\nu} F_{\mu\nu} + \overline{\psi} \gamma_\mu \partial_\mu \psi - m \overline{\psi} \psi. \tag{1}$$

Here the color indices associated with the gauge symmetry are suppressed. Of course we are really interested in the quantum theory, which requires definition with an ultraviolet regulator, such as the lattice. In this sense the coupling g and the mass m should be regarded as bare parameters. As the cutoff is removed, the bare parameters go to zero, as described by the simple renormalization group analysis reviewed in Section VI.

In this Lagrangian the change of variables

$$\psi \to e^{i\gamma_5\theta/2}\psi$$
$$\overline{\psi} \to \overline{\psi}e^{i\gamma_5\theta/2} \tag{2}$$

leaves the kinetic term for the fermions unchanged, but does not leave the mass term invariant

$$m\overline{\psi}\psi \to m\cos(\theta)\overline{\psi}\psi + im\sin(\theta)\overline{\psi}\gamma_5\psi = m_1\overline{\psi}\psi + m_5\overline{\psi}\gamma_5\psi. \tag{3}$$

So, if we allow a CP violating mass term of form $i\overline{\psi}\gamma_5\psi$, the theory appears to depend on three parameters g, m and θ or equivalently g, m_1 and m_5.

Introducing left and right fields

$$\psi_{R,L} = \frac{1}{2}(1 \pm \gamma_5)\psi \tag{4}$$

the rotated mass term is usually rewritten in terms of a complexification of the quark mass

$$\text{Re } m\overline{\psi}\psi + i\,\text{Im } m\sin(\theta)\overline{\psi}\gamma_5\psi = m\,\overline{\psi}_L\psi_R + m^*\,\overline{\psi}_R\psi_L. \tag{5}$$

Then the rotation of Eq. (2) corresponds to a change in the phase of m.

As we are just doing a change of variables, the physics of the classical theory should be independent of the phase of m. However, quantum mechanically this symmetry fails due to the non-invariance of the path integral measure. This is not entirely obvious since the determinant of the transformation factor $e^{i\gamma_5\theta/2}$ would appear to be unity since γ_5 is traceless. However all known regulators break this symmetry. How this physics manifests itself in a particular theoretical formalism depends on approach. With a Pauli-Villars regulator [4], the heavy auxiliary field has a phase in its mass; the theta parameter is the relative phase between the regulator mass and the physical fermion mass. In other continuum schemes the phase is often pushed into the path integral measure [5]. With Wilson fermions we have the added Wilson term, which is itself of a mass like form. Theta then becomes a relative phase between the Wilson term and the explicit mass [6]. With overlap fermions the chiral rotation involves two different versions of γ_5, one of which is not always traceless.

On renormalization, discussed in more detail in Section VI, these three parameters, $\{g,\,\text{Re } m,\,\text{Im } m\}$, are replaced by the overall scale of the strong interactions, usually called Λ_{qcd}, and renormalized values for the real and imaginary parts of the quark mass. Fig. (1) represents the resulting phase diagram in the latter two parameters.

Many discussions in the literature trade the imaginary part of the mass for a topological term $\tilde{F}_{\mu\nu}F_{\mu\nu}$ in the Lagrangian. This quantity is a total divergence, and its integral over all space-time is proportional to an integer winding number. The angle conjugate to this integer might be thought of as a further parameter. However, this variable and the phase of the mass are actually redundant since the anomaly allows one to rotate between them. Thus there is only one physically significant angle on which the theory depends. For this paper we adopt the convention that any topological term in the Lagrangian has been rotated into the phase of the mass.

After this rotation, traditional discussions use the magnitude and the phase of the mass as independent variables. However, we will see that because of ambiguities in defining the quark mass, the magnitude and phase are potentially singular coordinates. Thus it is cleaner to use as parameters the real and imaginary parts of the renormalized mass. This is an issue special to the one flavor theory; with multiple degenerate fermions, broken chiral symmetry with its attendant Goldstone bosons uniquely defines the massless theory.

III.   EFFECTIVE LAGRANGIANS

Outside of the lattice, effective chiral Lagrangians have long been among our most powerful tools to investigate non-perturbative phenomena. They build in the known chiral symmetries and have been highly successful in describing the physics of the light pseudoscalar mesons. Hints of the possibility for spontaneous CP violation in this approach go back some time [7]. Extensions to study the behavior with complex mass appear in several references [8–10]. In this section I will rehash some of these arguments in the context of the one flavor theory.

A quick but imprecise argument gives the expected picture. With only one flavor, there is only one light pseudo-scalar meson, referred to here as the η'. Were it not for anomalies, conventional chiral symmetry arguments suggest the mass squared of this particle would go to zero linearly with the quark mass,

$$m_{\eta'}^2 \sim m_q. \tag{6}$$

Throughout this paper we assume that appropriate powers of the strong interaction scale, Λ_{qcd} are inserted for dimensional purposes. But, just as the η' in the three flavor theory gets mass from anomalies, a similar contribution should appear here; assume it is simply an additive constant

$$m_{\eta'}^2 \sim m_q + c. \tag{7}$$

Consider describing this model by an effective potential for the η' field. This should include the possibility for these particles to self interact, suggesting something like

$$V(\eta') = \frac{m_q + c}{2}\eta'^2 + \lambda\eta'^4. \tag{8}$$

To get the phase diagram of Fig. (1), we add in a m_5 term as a linear piece in the eta prime field

$$V(\eta'; m_1, m_5) = \frac{m_1 + c}{2}\eta'^2 + \lambda\eta'^4 + m_5\eta'. \tag{9}$$

At $m_1 < -c$ the effective mass of the eta prime goes negative. This will give a spontaneous breaking in the canonical manner, with the field acquiring an expectation value

$$\langle\eta'\rangle \sim \langle\overline{\psi}\gamma_5\psi\rangle \sim \sqrt{\frac{|m_1| - c}{4\lambda}} \neq 0. \tag{10}$$

As this is a CP odd field, CP is spontaneously broken. In particular, vertices connecting odd numbers of physical mesons will not vanish, unlike in the unbroken theory where the number of pseudoscalar mesons is preserved modulo 2.

Note that this transition occurs at a negative quark mass, and nothing special happens at $m_1 = 0$. Of course the bare quark mass is not really physical since it is a divergent quantity in need of renormalization. Normally in the multiple flavor theory chiral symmetry forces this renormalization to be multiplicative, making vanishing mass special. However with only one flavor there is no chiral symmetry; thus, there is nothing to prevent an additive shift in this parameter. We will see later how such a shift is generated non-perturbatively by topologically non-trivial gauge configurations. Despite this, with a cutoff in place, these qualitative arguments suggest it is only for a negative quark mass that this parity violating phase transition will take place. And this also suggests that the magnitude of this gap is of order the square of the eta prime mass. As the latter is of order the strong interaction scale, the critical mass for the CP violating transition is of order Λ_{qcd}, as it must be.

This argument is suggestive but certainly not rigorous. To lend more credence to this qualitative picture, note that a similar phenomenon occurs in two dimensional electrodynamics. The Schwinger model is exactly solvable at zero bare mass, with the spectrum being a free massive boson. However, for negative bare mass, qualitative semi-classical arguments indicate the same structure as discussed above, with a spontaneous generation of a parity violating background electric field. Under the bosonization process [11, 12], the quark mass term corresponds to a sinusoidal term in the effective potential for the scalar field

$$m\overline{\psi}\psi \leftrightarrow \xi m\cos(2\sqrt{\pi}\eta') \tag{11}$$

6

where ξ is a numerical constant. Regularization and normal ordering are required for a proper definition but are not important here. Combining this with the photon mass from the anomaly suggests an effective potential for the η' field of form

$$V(\eta') \sim \frac{e^2}{2\pi}\eta'^2 - \xi m \cos(2\sqrt{\pi}\eta'). \tag{12}$$

For small positive m, the second term introduces multiple meson couplings, making the theory no longer free. It also shifts the boson mass

$$m_{\eta'}^2 \sim \frac{e^2}{\pi} + 4\pi\xi m \tag{13}$$

in a manner similar to Eq. (7). If the fermion mass is negative and large enough, the cosine term can dominate the behavior around small η', making the perturbative vacuum unstable. The bosonization process relates $\overline{\psi}\gamma_5\psi$ with $\sin(2\sqrt{\pi}\eta')$; thus, when η' gains an expectation value, so does the the pseudo-scalar density. Since the scalar field represents the electric field, this symmetry breaking represents the spontaneous generation of a background field. As discussed by Coleman [11, 12], this corresponds to a non-trivial topological term in the action, usually referred to as Θ.

Another way to understand the one flavor behavior is to consider several flavors and give all but one large masses. For example consider the three flavor case, and give two quarks a larger mass than the third. Model the light pseudoscalar sector of this theory with an effective field Σ taken from the group $SU(3)$. With two quarks of mass M and one of mass m, consider the potential

$$V(\Sigma) \propto -\text{ReTr}(\mathcal{M}\Sigma) \tag{14}$$

with mass matrix

$$\mathcal{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix}. \tag{15}$$

It is convenient to break this into two terms

$$V(\Sigma) \propto -\frac{M+m}{2}\text{ReTr}(\Sigma) + \frac{M-m}{2}\text{ReTr}(\Sigma h) \tag{16}$$

where

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{17}$$

When $M+m$ is positive The minimum of the first term in Eq. (16) occurs at the identity element. For the second term, note that the factor h of Σ is taken as an $SU(3)$ group element. The extrema

of this term occur when the product Σh is in the group center. For the case $M - m$ positive, there is a degenerate pair of minima occurring at

$$\Sigma = e^{\pm 2\pi i/3} h. \tag{18}$$

We see that Eq. (16) has two competing terms, one having a unique minimum at $\Sigma = I$ and the other having two degenerate ground states at the above complex values. For the degenerate case with $M = m$, only the first term is present and the vacuum is unique. However when $m = -M$ only the second term is present with its corresponding pair of degenerate vacua. Somewhere between these points must lie a critical value m_c where the situation shifts between a unique and a doubly degenerate vacuum.

To determine the critical mass, consider matrices of form

$$\Sigma = e^{i\phi \Gamma} \tag{19}$$

where

$$\Gamma = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

For these the potential is

$$V(\phi) \propto -m\cos(2\phi) - 2M\cos(\phi). \tag{21}$$

The extremum at $\phi = 0$ continuously changes from a minimum to a maximum at $m_c = -M/2$, the desired critical point. As discussed at the beginning of this section, it occurs at a negative value of m. The only dimensional scale present is M, to which the result must be proportional. This analysis immediately generalizes to larger flavor groups; for N_f flavors divided into a set of $N_f - 1$ of mass M and one of mass m we have $m_c = \frac{-M}{N_f - 1}$. As M becomes much larger than the scale of QCD, Λ_{qcd}, it is expected that the latter will replace M in setting the scale for the critical mass.

This discussion suggests that a similar phenomenon should occur on the lattice with one flavor of Wilson fermion. Here the bare mass is controlled by the hopping parameter. As the hopping parameter increases, the fermion mass decreases. In the plane of the gauge coupling and hopping parameter, a critical line should mark where the above parity breaking begins. In the lattice context the possibility of such a phase was mentioned briefly by Smit [13] and later extensively discussed by Aoki [14] and Aoki and Gocksch [15]. The latter papers also made some rather dramatic predictions for the breaking of both parity and flavor symmetries when more quark species are present.

8

Ref. [16] has argued that certain discrete symmetries such as parity cannot be spontaneously broken in vector-like theories such as QCD. This argument, however, assumes that parameters are taken such that the fermion determinant is positive. This is not true at negative quark mass, i.e. where these effective Lagrangian arguments suggest these strange phases exist. In more conventional language, the spontaneous symmetry breaking considered here occurs when the topological angle Θ takes the value π.

IV.   DIRAC EIGENVALUES

Amongst the lattice gauge community it has recently become quite popular to study the distributions of eigenvalues of the Dirac operator in the presence of the background gauge fields generated in simulations. There are a variety of motivations for this. First, in a classic work, Banks and Casher [17] related the density of small Dirac eigenvalues to spontaneous chiral symmetry breaking. Second, lattice discretizations of the Dirac operator based the Ginsparg-Wilson relation [18] have the corresponding eigenvalues on circles in the complex plane. The validity of various approximations to such an operator can be qualitatively assessed by looking at the eigenvalues. Third, using the overlap method [19] to construct a Dirac operator with good chiral symmetry has difficulties if the starting Wilson fermion operator has small eigenvalues. This can influence the selection of simulation parameters, such as the gauge action [20]. Finally, since low eigenvalues impede conjugate gradient methods, separating out these eigenvalues explicitly can potentially be useful in developing dynamical simulation algorithms [21]. This section is similar in philosophy to Ref. [9], wherein effective Lagrangian arguments similar to those in the previous section were used to make predictions for the eigenvalue structure for various numbers of flavors.

Despite this interest in the eigenvalue distributions, dangers lurk for interpreting the observations. Physical results come from the full path integral over both the bosonic and fermionic fields. Doing these integrals one at a time is fine, but trying to interpret the intermediate results is inherently dangerous. While the Dirac eigenvalues depend on the given gauge field, it is important to remember that in a dynamical simulation the gauge field distribution itself depends on the eigenvalues. This circular behavior gives a highly non-linear system, and such systems are notoriously hard to interpret.

To establish the context of the discussion. Consider a generic path integral for a gauge theory

$$Z = \int (dA)(d\psi)(d\overline{\psi}) \; e^{-S_G(A)+\overline{\psi}D(A)\psi}. \tag{22}$$

Here A and ψ represent the gauge and quark fields, respectively, $S_G(A)$ is the pure gauge part of the action, and $D(A)$ represents the Dirac operator in use for the quarks. As the action is quadratic in the fermion fields, a formal integration gives

$$Z = \int (dA) \; |D(A)| \; e^{-S_G(A)}. \tag{23}$$

Working on a finite lattice, $D(A)$ is a finite dimensional matrix, and for a given gauge field we can formally consider its eigenvectors and eigenvalues

$$D(A)\psi_i = \lambda_i \psi_i. \tag{24}$$

The determinant appearing in Eq. (23) is the product of these eigenvalues; so, the path integral takes the form

$$Z = \int (dA) \; e^{-S_G(A)} \prod_i \lambda_i. \tag{25}$$

Averaging over gauge fields defines the eigenvalue density

$$\rho(x+iy) = \frac{1}{NZ} \int (dA) \; |D(A)| \; e^{-S_G(A)} \sum_i \delta(x - \mathrm{Re}\lambda_i(A))\delta(y - \mathrm{Im}\lambda_i(A)). \tag{26}$$

Here N is the dimension of the Dirac operator, including volume, gauge, spin, and flavor indices.

In situations where the fermion determinant is not positive, ρ can be negative or complex; nevertheless, we still refer to it as a density. In addition, assume that ρ is real; situations where this is not true, such as with a finite chemical potential,[22] are fascinating but beyond the scope of this discussion.

At zero chemical potential, all actions used in practice satisfy "γ_5 Hermiticity"

$$\gamma_5 D \gamma_5 = D^\dagger. \tag{27}$$

With this condition, all non-real eigenvalues occur in complex conjugate pairs, implying for the density

$$\rho(z) = \rho(z^*). \tag{28}$$

This property will be shared by all the operators considered in the following discussion.

The quest is to find general statements relating the behavior of the eigenvalue density to physical properties of the theory. Repeating the earlier warning, ρ depends on the distribution of gauge fields A which in turn is weighted by ρ which depends on the distribution of A ....
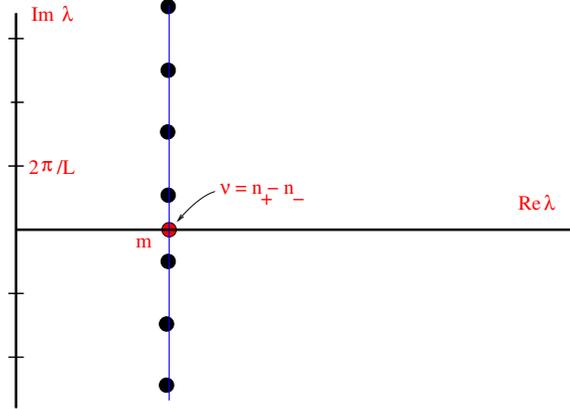
FIG. 2: In the naive continuum picture, all eigenvalues of the Dirac operator lie along a line parallel to the imaginary axis. In a finite volume these eigenvalues become discrete. The real eigenvalues divide into distinct chiralities and define a topological invariant.

A. The continuum

Of course the continuum theory is only really defined as the limit of the lattice theory. Nevertheless, it is perhaps useful to recall the standard picture, where the Dirac operator

$$D = \gamma_\mu(\partial_\mu + igA_\mu) + m$$

is the sum of an anti-Hermitian piece and the quark mass m. All eigenvalues have the same real part m

$$\rho(x+iy) = \delta(x-m)\tilde{\rho}(y).$$

The eigenvalues lie along a line parallel to the imaginary axis, while the Hermiticity condition of Eq. (27) implies they are either real or occur in complex conjugate pairs. Restricted to the subspace of real eigenvalues, γ_5 commutes with D and thus these eigenvectors can be separated by chirality. The difference between the number of positive and negative eigenvalues of γ_5 in this subspace defines an index related to the topological structure of the gauge fields [23]. The basic structure is sketched in Fig. (2).

It is useful to separately discuss the consequences of real versus complex eigenvalues. The former form a continuous distribution whereas the latter are all at the same point. This section concentrates on the density of complex eigenvalues, the next section turns to the consequences of the exactly real eigenvalues.

The Banks and Casher argument relates a non-vanishing $\tilde{\rho}(0)$ to the chiral condensate occurring
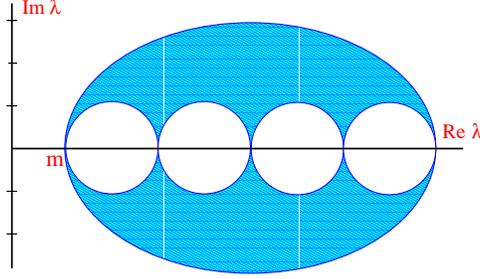
11

FIG. 3: Free Wilson fermions display an eigenvalue spectrum with a momentum dependent real part. This removes doublers by giving them a large effective mass.

when the mass goes to zero. We will say more on this later in the lattice context. Note that the naive picture suggests a symmetry between positive and negative mass. Due to anomalies, this is spurious. Indeed, with any number of flavors, flipping the sign of a single quark mass gives an inequivalent theory.

B. Wilson fermions

The lattice reveals the true intricacy of the situation arising from chiral anomalies. Without ultraviolet infinities, all naive symmetries of the lattice action are true symmetries. Naive fermions cannot have anomalies, which are cancelled by extra states referred to as doublers. Wilson fermions [24] avoid this issue by giving a large real part to those eigenvalues corresponding to the doublers. In particular, by modifying the hopping term of naive fermions, Wilson allowed the fermion mass to depend on momentum

$$m \rightarrow m + \frac{1}{a} \sum_{\mu} (1 - \cos(p_{\mu} a)) \tag{29}$$

thus giving the doublers a mass of order $1/a$. In momentum space, the free Wilson-Dirac operator takes the form

$$D_w = m + \frac{1}{a} \sum_{\mu} (i \sin(p_{\mu} a) \gamma_{\mu} + 1 - \cos(p_{\mu} a)). \tag{30}$$

The corresponding eigenvalue structure displays a simple pattern as shown in Fig. (3).

As the gauge fields are turned on, the eigenvalues shift around and blur this pattern. An additional complication is that the operator D is no longer normal, i.e. $[D, D^{\dagger}] \neq 0$ and the eigenvectors need not be orthogonal. The complex eigenvalues are still paired, although, as the gauge fields
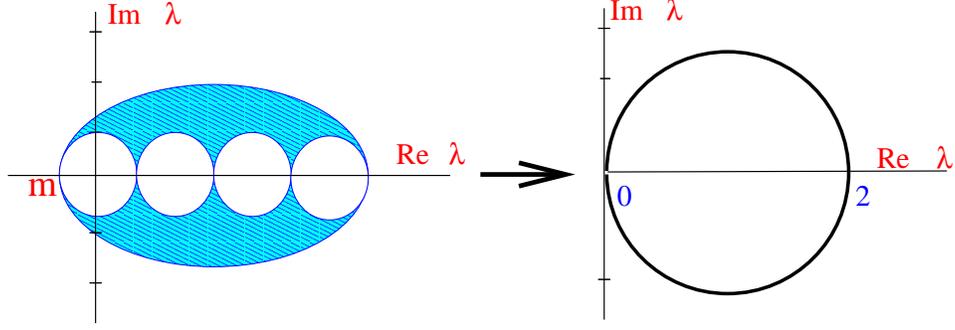
12

FIG. 4: The overlap operator is constructed by projecting the Wilson Dirac operator onto a circle.

vary, complex pairs of eigenvalues can collide and separate along the real axis. In general, the real eigenvalues will form a continuous distribution.

As in the continuum, an index can be defined from the spectrum of the Wilson-Dirac operator. Again, γ_5 Hermiticity allows real eigenvalues to be sorted by chirality. To remove the contribution of the doubler eigenvalues, select a point inside the leftmost open circle of Fig. (3). Then define the index of the gauge field to be the net chirality of all real eigenvalues below that point. For smooth gauge fields this agrees with the topological winding number obtained from their interpolation to the continuum. It also corresponds to the winding number discussed below for the overlap operator.

C. The overlap

Wilson fermions have a rather complicated behavior under chiral transformations. The overlap formalism[19] simplifies this by first projecting the Wilson matrix D_W onto a unitary operator

$$V = (D_W D_W^\dagger)^{-1/2} D_W. \tag{31}$$

This is to be understood in terms of going to a basis that diagonalizes $D_W D_W^\dagger$, doing the inversion, and then returning to the initial basis. In terms of this unitary quantity, the overlap Dirac operator is

$$D = 1 + V. \tag{32}$$

The projection process is sketched in Fig. (4). The mass used in the starting Wilson operator is taken to a negative value so selected that the low momentum states are projected to low eigenvalues, while the doubler states are driven towards $\lambda \sim 2$.

13

The overlap operator has several nice properties. First, it satisfies the Ginsparg-Wilson relation,[18]

$$\gamma_5 D + D \gamma_5 = D \gamma_5 D \tag{33}$$

which is now most succinctly written as the unitarity of V coupled with its γ_5 Hermiticity

$$\gamma_5 V \gamma_5 V = 1. \tag{34}$$

As it is constructed from a unitary operator, normality of D is guaranteed. But, most important, it exhibits a lattice version of an exact chiral symmetry [25]. The fermionic action $\overline{\psi} D \psi$ is invariant under the transformation

$$\begin{aligned} \psi &\to e^{i\theta \gamma_5} \psi \\ \overline{\psi} &\to \overline{\psi} e^{i\theta \hat{\gamma}_5} \end{aligned} \tag{35}$$

where

$$\hat{\gamma}_5 = -V \gamma_5. \tag{36}$$

As with γ_5, this quantity is Hermitean and its square is unity. Thus its eigenvalues are all plus or minus unity. The trace defines an index

$$\nu = \frac{1}{2} \mathrm{Tr} \hat{\gamma}_5 \tag{37}$$

which plays exactly the role of the index in the continuum. The factor of one half in this equation is due to the fact that the total number of real eigenvalues is even with each zero eigenvalue of D having a partner at $D = 2$, and both contribute to $\hat{\gamma}_5$.

Of course for the one flavor theory anomalies remove all traces of chiral symmetry; so, the use of the overlap operator in this case seems less motivated. Nevertheless, the formalism has the nice properties of having the Dirac operator be normal and of having exact zero modes. This allows an analysis of how, despite this apparent extra symmetry, the eigenvalue structure still permits the predicted smooth behavior for this theory around zero mass.

It is important to note that the overlap operator is not unique. Its precise form depends on the particular initial operator chosen to project onto the unitary form. Using the Wilson-Dirac operator for this purpose, the result still depends on the input mass used. From its historical origins in the domain wall formalism, this quantity is sometimes called the "domain wall height."

Because the overlap is not unique, an ambiguity can remain in determining the winding number of a given gauge configuration. Issues arise when $D_W D_W^\dagger$ is not invertible, and for a given gauge field this can occur at specific values of the projection point. This problem can be avoided for

14

"smooth" gauge fields. Indeed, an "admissibility condition," [26, 27] requiring all plaquette values to remain sufficiently close to the identity, removes the ambiguity. Unfortunately this condition is incompatible with reflection positivity [28]. Because of these issues, it is not known if the topological susceptibility is in fact a well defined physical observable. On the other hand, as it is not clear how to measure the susceptibility in a scattering experiment, there seems to be little reason to care if it is an observable or not.

To control issues related to exact zero modes, introduce a small mass and take the volume to infinity first and then the mass to zero. Toward this end, consider

$$\langle \overline{\psi}\psi \rangle = \langle \mathrm{Tr}\, D^{-1} \rangle = \left\langle \sum_i \frac{1}{\lambda_i + m} \right\rangle. \tag{38}$$

The signal for chiral symmetry breaking is a jump in this quantity as the mass passes through zero.

As the volume goes to infinity, replace the above sum with a contour integral around the overlap circle using $\lambda = 1 + e^{i\theta}$. Up to the trivial volume factor, one should evaluate

$$i \int_0^{2\pi} d\theta \frac{\rho(\theta)}{1 + e^{i\theta} + m}. \tag{39}$$

As the mass passes through zero, the pole at $\lambda = -m$ passes between lying outside and inside the circle, as sketched in Fig. (5). As it passes through the circle, the residue of the pole is $\rho(0) = \lim_{\theta \to 0} \rho(\theta)$. Thus the integral jumps by $2\pi\rho(0)$. This is the overlap version of the Banks-Casher relation [17]; a non-trivial jump in the condensate is correlated with a non-vanishing $\rho(0)$.

Taking the volume to infinity before taking the mass to zero is important here. On any finite volume the partition function is a finite and convergent integral which is analytic in the mass and there can be no phase transition.

Note that for multiple flavors the exact zero modes related to topology are suppressed by the mass and do not contribute to this jump. For one flavor, however, the zero modes do give rise to a non-vanishing but smooth contribution to the condensate, as discussed in the next section.[29]

There is an interesting contrast between the one flavor theory and the picture when there are multiple degenerate quarks. For example, with two flavors of light quarks one expects spontaneous chiral symmetry breaking. This is the explanation for the light mass of the pion, which is an approximate Goldstone boson. In the above picture, the two flavor theory should have a non-vanishing $\rho(0)$.

Now return to the one flavor theory. In this case there should be no chiral symmetry. The famous $U(1)$ anomaly breaks the naive symmetry. No massless physical particles are expected
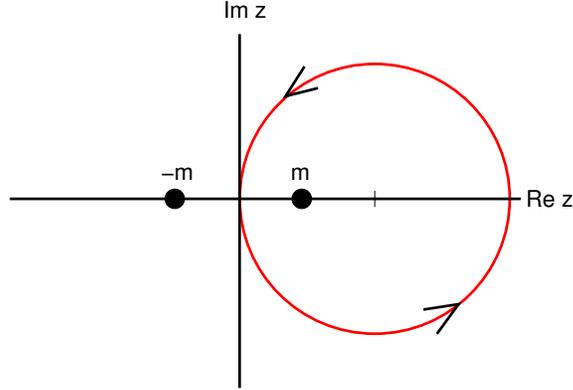
FIG. 5: As the mass changes sign a pole moves between inside and outside the overlap circle. This generates a jump in the condensate.

when the quark mass vanishes. Furthermore, the earlier simple chiral Lagrangian arguments [8, 30] indicate that no singularities are expected when a single quark mass passes through zero. Combined with the above discussion, this leads to the conclusion that for the one flavor theory $\rho(0)$ must vanish.

This leads us to an amusing paradox. Consider the original path integral after the fermions are integrated out. Changing the number of flavors N_f manifests itself in the power of the determinant

$$\int dA \; |D|^{N_f} \; e^{-S_g(A)}. \tag{40}$$

Naively this suggests that as you increase the number of flavors, the density of low eigenvalues should decrease. But we have just argued that with two flavors $\rho(0) \neq 0$ but with one flavor $\rho(0) = 0$. How can it be that increasing the number of flavors actually increases the density of small eigenvalues?

This is a clear example of how the non-linear nature of the problem can produce non-intuitive results. The eigenvalue density depends on the gauge field distribution, but the gauge field distribution depends on the eigenvalue density. It is not just the low eigenvalues that are relevant to the issue. Fermionic fields tend to smooth out gauge fields, and this process involves all scales. Smoother gauge fields in turn can give more low eigenvalues. Thus high eigenvalues influence the low ones, and this effect evidently can overcome the naive suppression from more powers of the determinant.

V. ZERO MODES

Through the index theorem, the topological structure of the gauge field manifests itself in zero modes of the massless Dirac operator. These are closely tied to the chiral anomaly and the behavior of the quark condensate for small quark masses. In this section I further explore this connection in the overlap formalism, concentrating on these zero modes.

As before, integrating out the fermionic fields from the path integral gives a determinant of the Dirac operator, D. For any given configuration of gauge fields this determinant is the product of the eigenvalues of this matrix. To control infrared issues, insert a small mass and write the path integral

$$Z = \int dA \, e^{-S_g} \prod_i (\lambda_i + m). \tag{41}$$

Here the λ_i are the eigenvalues of the kinetic part of the fermion determinant. If we take the mass to zero, any configurations which contain a zero eigenmode will have zero weight in the path integral. This suggests that for the massless theory we can ignore any instanton effects since those configurations don't contribute to the path integral.

What is wrong with this argument? The issue is not whether the zero modes contribute to the path integral, but whether they can contribute to physical correlation functions. To see how this goes, add some sources to the path integral

$$Z(\eta, \overline{\eta}) = \int dA \, d\psi \, d\overline{\psi} \, e^{-S_g + \overline{\psi}(D+m)\psi + \overline{\psi}\eta + \overline{\eta}\psi}. \tag{42}$$

Differentiation (in the Grassmann sense) with respect to η and $\overline{\eta}$ gives any desired fermionic correlation function. Now integrate out the fermions

$$Z = \int dA \, e^{-S_g - \overline{\eta}(D+m)^{-1}\eta} \prod_i (\lambda_i + m). \tag{43}$$

If we consider a source that overlaps with an eigenvector of D corresponding to one of the zero modes, i.e.

$$(\psi_0, \eta) \neq 0, \tag{44}$$

the source contribution introduces a $1/m$ factor. This cancels the m from the determinant, leaving a finite contribution as m goes to zero [29].

With multiple flavors, the determinant will have a mass factor from each. When several masses are taken to zero together, one will need a similar factor from the sources for each. This product

17

of source terms is the famous "'t Hooft vertex." [31, 32] While it is correct that instantons do drop out of Z, they survive in correlation functions.

While these issues are well understood theoretically, they can raise potential difficulties for numerical simulations. The usual procedure generates gauge configurations weighted as in the partition function. For a small quark mass, topologically non-trivial configurations will be suppressed. But in these configurations, large correlations can appear due to instanton effects. This combination of small weights with large correlations can give rise to large statistical errors, thus complicating small mass extrapolations. The problem will be particularly severe for quantities dominated by anomaly effects, such as the η' mass. A possible strategy to alleviate this effect is to generate configurations with a modified weight, perhaps along the lines of multi-canonical algorithms.[33]

In our case of the one flavor theory, the 't Hooft vertex is a quadratic form in the fermion sources. This will give a finite contribution to the condensate $\langle \overline{\psi}\psi \rangle$ that is continuous in the mass as the mass passes through zero. Note that unlike the jump generated from complex eigenvalues discussed in the previous section, this contribution remains even at finite volume. As the volume goes to infinity it is still only the one instanton sector that contributes since all instantons far from the source $\overline{\psi}\psi$ get suppressed by the mass factor.

Indeed, the 't Hooft vertex represents a non-perturbative additive shift to the quark mass [34]. As discussed in the next section, the size of this shift generally depends on scale and regulator details. Even with the Ginsparg-Wilson condition, the lattice Dirac operator is not unique, and there is no proof that two different forms have to give the same continuum limit for vanishing quark mass. Because of this, the concept of a single massless quark is not physical [3], invalidating one popular proposed solution to the strong CP problem. This ambiguity has been noted for heavy quarks in a more perturbative context [35] and is often referred to as the "renormalon" problem.

VI.   THE RENORMALIZATION GROUP

In previous sections we discussed the quark mass as a simple parameter without really defining it precisely. Because of confinement, quarks are not free particles and the usual definition of mass via particles propagating over long distances does not apply. Furthermore, as we are dealing with a quantum field theory, all bare parameters are divergent and need renormalization. In this section we use renormalization group methods to accomplish this, giving a precise definition to a quark

mass in the context of a given scheme. This will expose certain ambiguities in the mass definition. Most of this section is an expansion on the ideas presented in Ref. [3].

The renormalization process tunes all relevant bare parameters as a function of the cutoff while fixing a set of renormalized quantities. As we need to renormalize both the bare coupling and quark mass, we need to fix two physical observables. For this purpose, choose the lightest boson and the lightest baryon masses. As both are expected to be stable, this precludes any ambiguity from particle widths. As before, we denote the lightest boson as the η'. In the one flavor theory the lightest baryon p actually has spin 3/2 due to Pauli statistics, but for simplicity we still refer to it as the "proton." Because of confinement, the values of their masses are inherently non-perturbative quantities.

With the cutoff in place, the physical masses are functions of (g, m, a), the bare charge, the bare coupling, and the cutoff. For simplicity in this section, consider only real m and ignore the CP violating mass term m_5. Holding the masses constant, the renormalization process determines how g and m flow as the cutoff is removed. Because of asymptotic freedom, this flow eventually enters the perturbative regime and we have the famous renormalization group equations [36]

$$a\frac{dg}{da} \equiv \beta(g) = \beta_0 g^3 + \beta_1 g^5 + \dots \tag{45}$$

$$a\frac{dm}{da} \equiv m\gamma(g) = m(\gamma_0 g^2 + \gamma_1 g^4 + \dots) + \text{non-perturbative.} \tag{46}$$

The "non-perturbative" term should vanish faster than any power of the coupling. We include it explicitly in the mass flow because it will play a crucial role in the latter discussion. The values for the first few coefficients β_0, β_1, and γ_0 are known [37] and independent of regularization scheme.

The solution to these equations is well known

$$a = \frac{1}{\Lambda_{qcd}} e^{-1/2\beta_0 g^2} g^{-\beta_1/\beta_0^2} (1 + O(g^2)) \tag{47}$$

$$m = M g^{\gamma_0/\beta_0} (1 + O(g^2)). \tag{48}$$

In particular this shows how the bare coupling and bare mass are driven to zero as the cutoff is removed

$$g \sim \frac{1}{\log(1/a\Lambda_{qcd})} \tag{49}$$

$$m \sim \frac{1}{(\log(1/a\Lambda_{qcd}))^{\gamma_0/\beta_0}}. \tag{50}$$

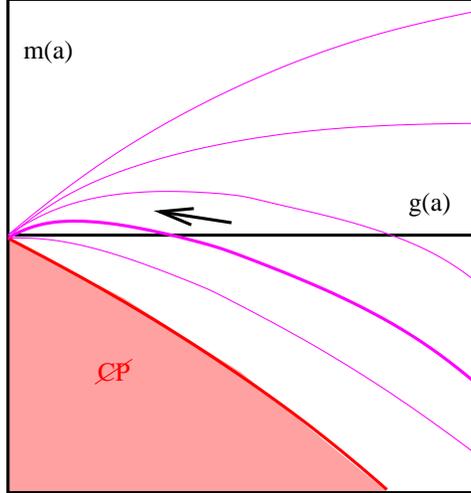This flow is sketched schematically in Fig. (6).

FIG. 6: As the cutoff is removed, the bare coupling and mass flow towards zero. Different flow lines correspond to different renormalized quark masses. With sufficiently negative mass one enters the regime of spontaneous CP violation. Anomaly effects preclude the line of vanishing bare quark mass from being renormalization group invariant.

The quantities Λ_{qcd} and M are integration constants for the renormalization group equations. We refer to Λ_{qcd} as the overall strong interaction scale and M as the renormalized quark mass. Their values depend on the explicit renormalization scheme as well as the physical values being held fixed in the renormalization process, i.e. the proton and η' masses. This connection is highly non-perturbative. Indeed the particle masses being fixed are long distance properties, and thus are tied to the renormalization group flow far out of the perturbative regime.

Turning things around, we can consider the physical particle masses as functions of these integration constants. Simple dimensional analysis tells us that the dependence of physical masses must take the form

$$m_p = \Lambda_{qcd} H_p(M/\Lambda_{qcd}) \tag{51}$$

$$m_{\eta'} = \Lambda_{qcd} H_{\eta'}(M/\Lambda_{qcd}) \tag{52}$$

where the $H_i(x)$ are dimensionless functions whose detailed form is highly non-perturbative.

For the case of multiple degenerate quark flavors we expect the square of the lightest boson mass to vanish linearly as the renormalized quark mass goes to zero. This means we anticipate a square root singularity in the corresponding $H(x)$ at $x = 0$. Indeed, requiring the singularity to occur at the origin removes any additive non-perturbative ambiguity in defining the renormalized

20

mass.

For the one flavor theory, things are more subtle. As discussed above, we expect physics to behave smoothly as the quark mass passes through zero. That is, we do not expect $H_i(x)$ to display any singularity at $x = 0$. Non-perturbative dynamics generates an additional contribution to the mass of the pseudo-scalar meson; thus, the $M = 0$ flow generically corresponds to a positive value of $m_{\eta'}$. While a $m_{\eta'} = 0$ flow line can exist, it represents the boundary of the CP violating phase discussed earlier and has little to do with massless quarks.

Now we come to the question of scheme dependence. Given some different renormalization prescription, i.e. a modified lattice action, the precise flows will change. Although the behavior dictated in Eqs. (47,48) must be preserved, the integration constants (Λ_{qcd}, M) and the functions $H_i(x)$ will in general be modified. Marking the new quantities with tilde's, matching the schemes to give the same physics requires

$$m_i = \Lambda_{qcd} H_i(M/\Lambda_{qcd}) = \tilde{\Lambda}_{qcd} \tilde{H}_i(\tilde{M}/\tilde{\Lambda}_{qcd}). \tag{53}$$

Upon the removal of the cutoff, two different valid schemes should give the same result for the physical masses. An important distinction for the one flavor theory, without chiral symmetry to protect things, is the absence of any reason for the vanishing of M to require the vanishing of $\tilde{M}$.

On changing schemes, we introduce new definitions for the coupling and mass. To match onto the perturbative limit, it is reasonable to restrict these definitions to agree at leading order. Thus we should require

$$\tilde{g} = g + O(g^3) \tag{54}$$
$$\tilde{m} = m(1 + O(g^2)) + \text{non-perturbative}. \tag{55}$$

Here the "non-perturbative" terms should vanish faster than any power of the coupling, but are not in general proportional to m. In particular, a non-perturbative additive shift in the quark mass follows qualitatively from the analysis of zero modes in the previous section.

The requirements for the perturbative limit apply at fixed cutoff. Indeed, the interplay of the $a \to 0$ and the $g \to 0$ limits is rather intricate. As $g \to 0$ at fixed a the quarks decouple and we have a theory of free quarks and gluons. The limit $a \to 0$ at fixed g brings on the standard divergences of relativistic field theory. The proper continuum limit follows the renormalization group trajectory with both a and g going together in the appropriate way and gives a theory where important non-perturbative effects such as confinement are relevant.

Assuming only the matching conditions in Eq. (54,55) leaves the freedom to do some amusing things. As a particularly contrived example, consider

$$\tilde{g} = g \tag{56}$$

$$\tilde{m} = m - Mg^{\gamma_0/\beta_0} \times \frac{e^{-1/2\beta_0 g^2} g^{-\beta_1/\beta_0^2}}{\Lambda_{qcd} a}. \tag{57}$$

The last factor vanishes than any power of g, but is crafted to go to unity along the renormalization group trajectory. Note that a power of the scale factor as inserted here is necessary for non-perturbative phenomena to be relevant to the continuum limit [31]. With this form, one can immediately relate the old and new renormalized masses

$$\tilde{M} \equiv \lim_{a \to 0} \tilde{m} \tilde{g}^{-\gamma_0/\beta_0} = M - M = 0. \tag{58}$$

Thus for any M, another scheme always exists where the renormalized quark mass vanishes. The possibility of such a transformation demonstrates that masslessness is not a physical concept for the one flavor theory, or more generally for a non-degenerate quark in a multi-flavor theory.

I close this section with some remarks on trying to define the quark mass is through the operator product expansion. Expanding the product of two electromagnetic currents at small separations will bring in a variety of quark operators. Among them is the simple scalar combination $\overline{\psi}\psi$. The short distance behavior involves a triangle diagram which by γ_5 symmetry should vanish when the quark mass does. Thus one might define the zero mass theory where the coefficient of the singular part of this term in the operator product expansion has a zero.

This approach raises several issues. Because the 't Hooft vertex in the one flavor theory takes the same form $\overline{\psi}\psi$, it clouds the definition of this as a renormalized operator. The same scheme dependent additive shift that plagues the quark mass can modify where this zero occurs. Furthermore, the additive shift in the quark mass makes it unclear whether this definition of vanishing quark mass has any connection with the quark masses in some particular effective chiral Lagrangian. It is even possible that the zero mass theory defined this way may be in the CP violating phase, in which case it certainly doesn't represent something physically interesting. Indeed, it is unclear what experiment if any could determine if the quark mass defined via the operator product expansion vanishes.

VII.   MORE COLORS AND HIGHER REPRESENTATIONS

A popular approximation considers QCD in the limit of a large number of colors, replacing the $SU(3)$ gauge group with $SU(N_c)$. In the limit $N_c \to \infty$ planar diagrams dominate and internal quark loops are suppressed [38]. Unfortunately, most of the effects being discussed in this paper are suppressed as well, being higher order corrections in the $1/N_c$ expansion. Nevertheless, an early indication of the basic one flavor behavior came during a study of the large N_c limit [8]. Thus it seems reasonable that for larger but finite N_c and retaining the fundamental representation of the gauge group for the fermions, we should have a similar structure to that of Fig. (1).

Recently a rather interesting variation of the large number of colors expansion has been proposed [39–41]. Rather than taking the quarks in the fundamental representation, they use the antisymmetric tensor product of two fundamental representations. For the case $N_c = 3$ these theories are equivalent, since it is a convention whether quarks are in the 3 or the $\overline{3}$ representation of $SU(3)$. When N_c increases, however, the product representation is larger, of dimension $N_c(N_c-1)/2$ rather than N_c, and enhances the effects of quark loops. As the number of colors goes to infinity the distinction between the antisymmetric and symmetric tensor product becomes unimportant and the papers in Ref. [40] have gone on to make inferences between the these theories at $N_c = \infty$ and the bosonic sector of supersymmetric Yang-Mills theory. Thus they suggested using this variation on the large color expansion to extract information about one flavor QCD.

Working with fermions in a representation other than the fundamental modifies the effect of topological structures. In particular, although there still is no continuous chiral symmetry in the one flavor theory, certain discrete chiral symmetries can arise. It is then natural to ask if these discrete symmetries could be spontaneously broken. Ref. [40] suggests that they are, and $\langle \overline{\psi}\psi \rangle$ represents an order parameter for this breaking. Evaluating this in the large N_c limit, they propose that this might give some approximate information on the theory with a smaller number of colors. Of course for the three color case of interest, there is no such extra symmetry to be broken and it makes no sense to consider $\langle \overline{\psi}\psi \rangle$ as an order parameter in the traditional sense. But these discrete symmetries are interesting in their own right and it is perhaps instructive to ask if they are indeed broken spontaneously for larger N_c.

The extra symmetries arise in cases where the zero modes of the Dirac operator are automatically degenerate. This is the case for the gauge group $SU(N)$ for $N > 3$ and fermions in the antisymmetric tensor product of two fundamental representations. Consider an instanton config-

uration, and rotate it to appear in the $SU(2)$ subgroup involving only the first two colors. If we break the antisymmetric representation into multiplets under this $SU(2)$ subgroup, it will consist of one singlet from both indices being in the first two colors, $N-2$ doublets with only one index from the first two and finally $(N-2)*(N-3)/2$ additional singlets involving only the higher values for the indices. From the instanton we expect one zero mode for each of the doublets, giving $N-2$ overall. From the earlier discussion, this means the 't Hooft vertex will involve the product of $N-2$ fermion bilinears.

Now consider the basic $U(1)$ chiral rotation of Eq. (2). The 't Hooft vertex still violates this symmetry; however, if the angle θ is chosen to be a multiple of $\frac{2\pi}{N_c-2}$, this vertex remains invariant. Thus this theory has a hidden discrete Z_{N_c-2} chiral symmetry. The conventional angle Θ conjugate to the gauge field topology differs from the phase of the mass term by a factor of $N_c - 2$.

This discussion assumes that all anomaly effects arise through topological structures and the 't Hooft vertex. On the lattice one might also expect lattice artifacts to break the symmetry, much as they do for conventional chiral symmetry in the Wilson action. Even for the overlap, there can be rough gauge configurations on which the winding number is ill defined. We assume that these issues involve higher dimensional "irrelevant" operators and they disappear in the continuum limit.

Turning on a complex quark mass, the theory is invariant under multiplication of this mass by an element of Z_{N_c-2}. Thus the phase diagram of Fig. (1) must be modified to incorporate this symmetry. To be specific, consider $SU(5)$ with the fermions in the 10 dimensional representation. (I skip over $SU(4)$ with fermions in the 6 dimensional representation to avoid the complication of baryons being bosons made up of only two quarks.) By the above arguments the five color theory has a Z_3 discrete chiral symmetry. One simple modification of the phase diagram for complex mass that incorporates this symmetry is sketched in Fig. (7).

As drawn, this figure assumes that this Z_3 symmetry is unbroken. An alternative possibility is that it is spontaneously broken. In this case the three transition lines would extend to the origin. The order parameter for these transition lines is the expectation of the η' field, which should undergo a finite jump as one passes through them. Given the highly suppressed contributions of instantons in this theory, and in light of the smooth behavior of the three color theory when the mass vanishes, this seems rather unmotivated at small N_c, but the possible alternate phase diagram is sketched in Fig. (8). It appears possible that at some large but finite N_c there is a change in behavior from that exemplified by a unique vacuum and transitions not reaching the origin, as in Fig. 7, to the case where there are $N_c - 2$ degenerate vacua at vanishing mass as in Fig. 8.
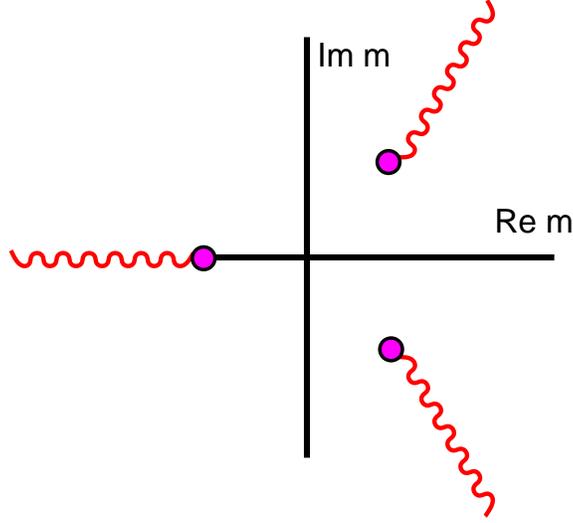
FIG. 7: A possible phase diagram for one flavor quark-gluon dynamics with gauge group $SU(5)$ and the fermions in the 10 representation. Unlike the $SU(3)$ case, there are now three first order phase transitions all pointing at the origin. The transition along the negative real axis represents a spontaneous breaking of CP symmetry. As the number of colors increases, additional transition lines should appear, with endpoints converging to the origin at $N_c = \infty$ where there is a spontaneously broken $U(1)$ symmetry.

Whether the discrete Z_{N_c-2} symmetry is broken or not, the point where the mass vanishes is now well defined, as long as one uses a regulator which respects this discrete symmetry. At this point $\langle \overline{\psi} \psi \rangle$ either vanishes or shows first order jumps to the other phases. This contrasts with the usual QCD case with gauge group $SU(3)$ where there is no residual discrete symmetry and this "condensate" has a smooth and non-vanishing behavior. As the mechanism for generating this expectation is rather different in the $N_c = 3$ case from the spontaneous symmetry breaking for more colors, it is unclear whether there should be any numerical connection [42].

As N_c increases, there should be a corresponding growth in the number of transition lines in the complex mass plane. In terms of the usual angle Θ appearing for topologically non-trivial gauge configurations, these transitions are all equivalent and represent $\Theta = \pi$. The convergence of these lines towards the origin can potentially give rise to the spontaneously broken $U(1)$ chiral symmetry expected in the $N_c \to \infty$ limit.

Note that this situation contrasts sharply with the behavior of QCD with several degenerate flavors. There it is also true that the topologically defined phase Θ differs by an integer factor from the phase of the quark mass. In the complex mass plane there are also expected to be several
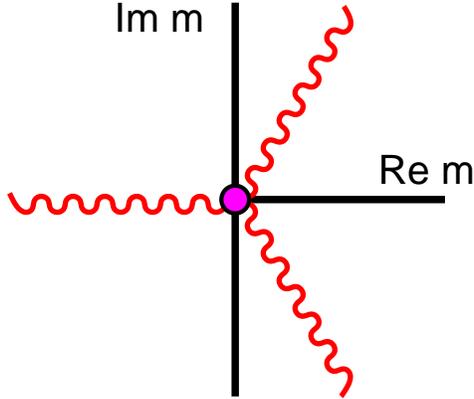
FIG. 8: An alternative possible phase diagram for one flavor quark-gluon dynamics with gauge group $SU(5)$ and the fermions in the 10 representation. This represents the situation where the three first order transitions meet at a triple point at the origin. In this case the discrete Z_3 chiral symmetry is spontaneously broken.

transition lines converging on the origin [1, 43]. However in this case there are massless Goldstone bosons when the mass exactly vanishes.

Considering other higher fermion representations, similar discrete symmetries are expected, such as a Z_5 for color $SU(3)$ with fermions in the 6 representation. For the adjoint case, each zero mode is $2N_c$ degenerate and we have a discrete Z_{2N_c} chiral symmetry, although the meaning of confinement in this theory is obscured since gluons can screen individual quarks. Going still further, one has to worry about whether one enters a conformal phase and/or asymptotic freedom is lost.

VIII. CONCLUSIONS

One flavor QCD is a fascinating system. Chiral symmetry, which is so crucial to our conventional understanding of the strong interactions, plays a rather strange role here. Indeed, anomalies mean that the naive classical chiral symmetry must disappear from the problem. This paper has discussed the qualitative behavior of the one flavor theory as a function of the quark mass. As summarized in Fig. (1), a second order phase transition is expected at non-zero negative mass. At this point the η' mass vanishes, while for still more negative mass this field acquires an expectation value, marking a CP violating phase.

This picture enables partial answers to many of the questions raised in the introduction. Indeed, chiral symmetry is in some sense irrelevant to the one flavor theory. Physics varies smoothly

and continuously for small masses and the location of the $m = 0$ point is not well defined. The quark condensate $\langle \overline{\psi}\psi \rangle$ is automatically non-zero and ceases to be a natural order parameter for any broken symmetry. However, with more colors and quarks in higher representations than the fundamental, discrete chiral symmetries can emerge for which the condensate may be an order parameter. However, it is an open question when these symmetries are expected to be broken for a finite number of colors.

Many of these details are in principle amenable to study in numerical simulations. Such simulations are made more difficult by the small mass region, and involve sign problems when the quark mass is negative. The latter will become particularly severe near the phase of spontaneously broken CP. Nevertheless, the absence of massless Goldstone bosons should alleviate these problems relative to theories with more flavors.

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