Anomalies and chiral symmetry in QCD

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Abstract

I review some aspects of the interplay between anomalies and chiral symmetry. The quantum anomaly that breaks the U(1) axial symmetry of massless QCD leaves behind a flavor-singlet discrete chiral invariance. When the mass is turned on this residual symmetry has a close connection with the strong CP violating parameter theta. One result is that a first order transition is usually expected when the strong CP violating angle passes through pi. This symmetry can be understood either in terms of effective chiral Lagrangians or in terms of the underlying quark fields.

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1. Introduction

The classical Lagrangian for QCD couples left and right handed quark fields only through mass terms. Thus naively the massless theory will have independent conserved currents associated with each handedness. For N_f massless flavors, this would be an independent $U(N_f)$ symmetry associated with each chirality, giving a full symmetry that is often written in terms of axial and vector fields as $U(N_f)_V \times U(N_f)_A$. As is well known, this full symmetry does not survive quantization, being broken to a $SU(N_f)_V \times SU(N_f)_A \times U(1)_B$, where the $U(1)_B$ represents the symmetry of baryon number conservation. The only surviving axial symmetries of the quantum theory are non-singlet under flavor symmetry.

This breaking of the classical $U(1)$ axial symmetry is tied to the possibility of introducing into massive QCD a CP violating parameter, usually called Θ. For an extensive recent review of this quantity, see Ref. [1]. While such a term is allowed from fundamental principles, experimentally it appears to

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be extremely small. This raises an unresolved puzzle for attempts to unify the strong interactions with the weak. Since the weak interactions do violate CP, why is there no residue of this remaining in the strong sector below the unification scale?

One goal here is to provide a qualitative picture of the Θ parameter in meson physics. I will concentrate on symmetry alone and will not attempt to rely on any specific form for an effective Lagrangian. I build on a connection between Θ and a flavor-singlet Z_{N_f} symmetry that survives the anomaly. This symmetry predicts that, if the lightest quarks are massive and degenerate, then a first order transition is expected when Θ passes through π. This transition is quite generic, and only can be avoided under limited conditions with one quark considerably lighter than the others. It will also become clear why the sign of the quark mass is relevant for an odd number of flavors, an effect unseen in naive perturbation theory.

This picture has evolved over many years. The possibility of the spontaneous CP violation occurring at $\Theta = \pi$ is tied to what is known as Dashen's phenomenon [2], first noted even before the days of QCD. In the mid 1970's, 't Hooft [3] elucidated the underlying connection between the chiral anomaly and the topology of gauge fields. Later Witten [4] used large gauge group ideas to discuss the behavior at $\Theta = \pi$ in terms of effective Lagrangians. Ref. [5] lists a few of the early studies of the effects of Θ on effective Lagrangians via a mixing between quark and gluonic operators. The topic continues to appear in various contexts; for example, Ref. [6] contains a different approach to understanding the transition at $\Theta = \pi$ via the framework of the two-flavor Nambu Jona-Lasinio model.

I became interested in the issues while trying to understand difficulties with formulating chiral symmetry on the lattice. Much of the picture presented here is implicit in my 1995 paper on quark masses [7]. Since then the topic has become highly controversial, with the realization of ambiguities precluding a vanishing up quark mass solving the strong CP problem [8] and the appearance of an inconsistency with one of the popular algorithms in lattice gauge theory [9]. Despite the controversies, both are direct consequences of the interplay of the anomaly and chiral symmetry discussed here. The fact that these issues remain so disputed is what has driven me to write this overview.

Section 2 reviews the conventional picture of spontaneous chiral symmetry breaking wherein the light pseudoscalars are identified as approximate Goldstone bosons in an effective Lagrangian. Here I introduce a mass contribution, coming from the anomaly, for for the flavor singlet pseudoscalar meson. Section 3 reviews how the anomaly arises in terms of the underlying quark fields and gauge fields with non-trivial topology. Here the fact that the theory requires a regulator that breaks chiral symmetry is crucial. Section 4 returns to the effective field picture and exposes the flavor singlet Z_{N_f} symmetry. In Section 5 I add in a small quark mass to break the chiral symmetry. Depending on the mass and N_f, the effective potential can display multiple meta-stable minima. Doing an anomalous chiral rotation on the mass term brings in the parameter Θ. The first order transition at $\Theta = \pi$ corresponds to a jump of the physical vacuum between two distinct degenerate minima. At this point the theory spontaneously breaks CP invariance. Section 6 discusses increasing the mass of one quark species to allow an interpolation between various N_f. Section 7 draws concluding remarks.

I start with a few reasonably uncontroversial assumptions. First QCD with N_f light quarks should exist as a field theory and exhibit confinement in the usual way. I assume the validity of the standard picture of chiral symmetry breaking involving a quark condensate $\langle\overline{\psi}\psi\rangle \neq 0$. The conventional chiral perturbation theory based on expanding in masses and momenta around the chiral limit should make sense. I assume the usual result that the anomaly generates a mass for the η' particle with this mass surviving in the chiral limit. And I consider N_f small enough to avoid any potential conformal phase of QCD [10].

Throughout I use the language of continuum field theory. I do have in mind that some non-perturbative regulator has been imposed to define various products of fields, such as the condensing combination $\sigma = \overline{\psi}\psi$. For a momentum space cutoff, I assume that it is much larger than Λ_{QCD}. Correspondingly, for a lattice cutoff, then I imagine that the lattice spacing is much smaller than $1/\Lambda_{QCD}$. Thus I ignore any lattice artifacts that are expected to vanish in the continuum limit.

2. Effective potentials

I begin by considering an effective potential V as a function of the various meson fields in the problem. Intuitively, V represents the energy of the lowest state for a given field expectation. More formally, this can be defined in the standard way via a Legendre transformation. I will ignore the well known result that effective potentials must be convex functions of their arguments. This is easily understood in terms of a Maxwell construction involving the phase separation that will occur if one asks for a field expectation in what would otherwise be a concave region. Ignoring this effect allows me to use the normal language of spontaneous symmetry breaking corresponding to having an effective potential with more than one minimum. When the underlying theory possesses some symmetry but the individual minima do not, spontaneous breaking comes about when the vacuum selects one minimum arbitrarily.

I frame the discussion in terms of composite scalar and pseudoscalar fields

$$\begin{array}{ccc} \sigma & \sim & \overline{\psi}\psi \\ \pi_\alpha & \sim & i\overline{\psi}\lambda_\alpha\gamma_5\psi \\ \eta' & \sim & i\overline{\psi}\gamma_5\psi \\ \delta_\alpha & \sim & \overline{\psi}\lambda_\alpha\psi \end{array} \tag{1}$$

Here the λ_α are the generalization of the usual Gell-Mann matrices to $SU(N_f)$. The δ field is listed here for completeness, but will not play a role in the following discussion. As mentioned earlier, I assume some sort of regulator, perhaps a lattice, is in place to define these products of fields at the same point.

Initially consider degenerate quarks with a small common mass m. I also begin by restricting N_f to be even, returning later to the subtleties arising for an odd number of flavors. And, as mentioned earlier, I keep N_f small enough to maintain asymptotic freedom as well as to avoid any possible conformal phases.

The conventional picture of spontaneous chiral symmetry breaking at $m = 0$ begins with the vacuum acquiring a quark condensate with $\langle\overline{\psi}\psi\rangle = \langle\sigma\rangle = v \neq 0$. In terms of the effective potential, $V(\sigma)$ should acquire a double well structure, as sketched in Fig. 1. The symmetry under $\sigma \leftrightarrow -\sigma$ is associated with the invariance of the action under a flavored chiral rotation. For example, with two flavors the change of variables

$$\begin{array}{l} \psi \longrightarrow e^{i\pi\tau_3\gamma_5/2}\psi \\ \overline{\psi} \longrightarrow \overline{\psi}e^{i\pi\tau_3\gamma_5/2} \end{array} \tag{2}$$

leaves the massless action invariant but changes the sign of σ. Here τ_3 is the conventional Pauli matrix corresponding to the third component of isospin.

Extending the effective potential to a function of the non-singlet pseudoscalar fields gives the standard picture of Goldstone bosons. These are massless when the quark mass vanishes, corresponding to $N_f^2 - 1$ "flat" directions for the potential. It is useful to introduce the $SU(N_f)$ valued effective

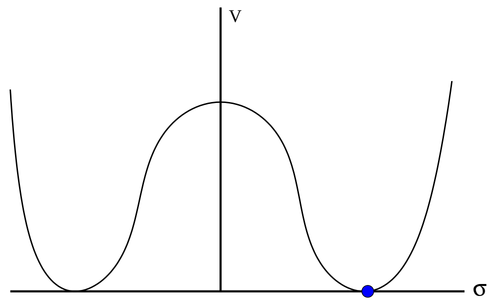

Figure 1: Spontaneous chiral symmetry breaking is represented by a double well effective potential with the vacuum settling into one of two possible minima. In this minimum chiral symmetry is broken by the selection of a specific value for the quark condensate.

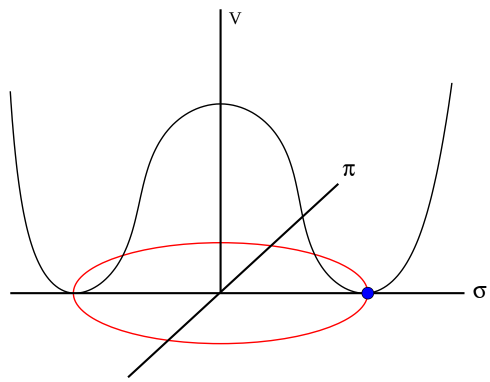

Figure 2: The flavor non-singlet pseudoscalar mesons are Goldstone bosons corresponding to flat directions in the effective potential.

field

$$\Sigma = e^{i\lambda_\alpha \pi_\alpha / F_\pi} \sim \overline{\psi}_L \psi_R \tag{3}$$

where the matrices λ generalize of the Gell-mann matrices to $SU(N_f)$, flavor indices that make this a matrix quantity are suppressed, and the pion decay constant F_π is inserted as a convenient normalization but will play no role in the qualitative discussion here. The left and right Fermi fields are defined as usual by $\psi_{L,R} = \frac{1 \mp \gamma_5}{2} \psi$. In terms of Σ the chiral invariance of the potential takes the simple form

$$V(\Sigma) = V(g_L^\dagger \Sigma g_R) \tag{4}$$

where g_L and g_R are arbitrary elements of $SU(N_f)$. One flat direction is sketched in Fig. 2.

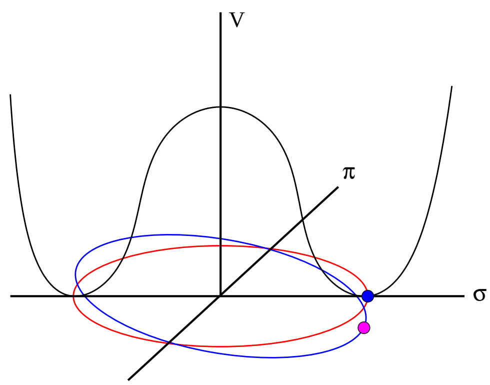

Figure 3: A small quark mass term tilts the effective potential, selecting one direction for the true vacuum and giving the Goldstone bosons a mass.

The introduction of a small mass for the quarks effectively tilts the potential $V(\sigma) \to V(\sigma) - m\sigma$. This selects one of the minima as the true vacuum, driving the above matrix $\Sigma \to I$. The tilting of the potential breaks the global symmetry and gives the Goldstone bosons a small mass proportional to the square root of the quark mass, as sketched in Fig. 3

This picture is, of course, completely standard. It is also common lore that the anomaly prevents the η' from being a Goldstone boson and leaves it with a mass of order Λ_{QCD} even in the massless quark limit. The effective potential V must not be symmetric under the following anomalous rotation by an angle ϕ

$$\begin{array}{rl} \sigma \to & \sigma\cos(\phi) + \eta' \sin(\phi) \\ \eta' \to & -\sigma\sin(\phi) + \eta' \cos(\phi). \end{array} \tag{5}$$

If we consider the effective potential as a function of the fields σ and η', it should have a minimum at $\sigma \sim v$ and $\eta' \sim 0$. Expanding about that point we expect a qualitative form

$$V(\sigma, \eta') \sim m_\sigma^2(\sigma - v)^2 + m_{\eta'}^2 \eta'^2 + O((\sigma - v)^3, \eta'^4) \tag{6}$$

where both m_σ and $m_{\eta'}$ remain of order Λ_{QCD}, even in the chiral limit. And, at least with an even number of flavors as considered here, there should be a second minimum with $\sigma \sim -v$. Expanding about this point gives the

$$V(\sigma, \eta') \sim m_\sigma^2(\sigma + v)^2 + m_{\eta'}^2 \eta'^2 + O((\sigma + v)^3, \eta'^4). \tag{7}$$

At this point one can ask whether we know anything else about the effective potential in this (σ, η') plane. In section 4 I show that indeed we do, and the potential has a total of N_f equivalent minima in the chiral limit. But first I digress to review how the above minima arise in quark language.

3. Quark fields

The classical QCD Lagrangian has a symmetry under a rotation of the underlying quark fields

$$\begin{aligned} \psi &\to e^{i\phi\gamma_5/2}\psi \\ \overline{\psi} &\to \overline{\psi}e^{i\phi\gamma_5/2} \end{aligned} \tag{8}$$

This corresponds directly to the transformation of the composite fields given in Eq. 5. This symmetry is "anomalous" and thus any regulator must break it with a remnant surviving in the continuum limit. The specifics of how this works depend on the details of the regulator, but a simple understanding [11] comes from considering the fermionic measure in the path integral. If we make the above rotation on the field ψ, the measure changes by the determinant of the rotation matrix

$$d\psi \to |e^{-i\phi\gamma_5/2}|d\psi = e^{-i\phi \mathrm{Tr}\gamma_5/2}d\psi. \tag{9}$$

Here the subtlety of the regulator comes in. Naively γ_5 is a simple four by four traceless matrix. If it is indeed traceless, then the measure would be invariant. However in the regulated theory this is not the case. This is intimately tied with the index theorem for the Dirac operator in topologically non-trivial gauge fields.

A typical Dirac action takes the form $\overline{\psi}(D+m)\psi$ with D a function of the gauge fields. In the naive continuum theory D is anti-Hermitian, $D^\dagger = -D$, and anti-commutes with γ_5, i.e. $[D,\gamma_5]_+ = 0$. What complicates the issue with fermions is the well known index theorem: if a background gauge field has winding ν, then there are known to be at least ν exact zero eigenvalues of D. Furthermore, on the space spanned by the corresponding eigenvectors, γ_5 can be simultaneously diagonalized with D. The net winding number equals the number of positive eigenvalues of γ_5 minus the number of negative eigenvalues. This theorem is well known and well reviewed elsewhere [12]. Here I only will use the fact that in this subspace the trace of γ_5 does not vanish, but equals ν.

What about the higher eigenvalues of D? Because $[D,\gamma_5]_+ = 0$, these appear in opposite sign pairs; i.e. if $D|\psi\rangle = \lambda|\psi\rangle$ then $D\gamma_5|\psi\rangle = -\lambda\gamma_5|\psi\rangle$. For an anti-Hermitean D, these modes are orthogonal with $\langle\psi|\gamma_5\psi\rangle = 0$.

As a consequence, γ_5 is traceless on the subspace spanned by each pair of eigenvectors.

So what happened to the opposite chirality states to the zero modes? In a regulated theory they are in some sense "above the cutoff." In a simple continuum discussion they have been "lost at infinity." With a lattice regulator there is no "infinity"; so, something more subtle must happen. With the overlap[13] or Wilson[14] fermions, one gives up the anti-Hermiticity of D. Most eigenvalues still occur in conjugate pairs and do not contribute to the trace of γ_5. However, in addition to the small real eigenvalues representing the zero modes, there are additional modes where the eigenvalues are large and real. With Wilson fermions these appear as massive doubler states. With the overlap, the eigenvalues are constrained to lie on a circle. In this case, for every exact zero mode there is another mode of opposite chirality lying on the opposite side of the circle. These modes are effectively massive and break chiral symmetry.

So with the regulator in place, the trace of γ_5 does not vanish on gauge configurations of non-trivial topology. The change of variables indicated in Eq. 9 introduces into the path integral a modification of the weighting by a factor

$$e^{-i\phi \mathrm{Tr}\gamma_5} = e^{-i\phi N_f \nu} \tag{10}$$

here I consider applying the rotation to all flavors equally, thus the factor of N_f in the exponent. The conclusion is that gauge configurations that have non-trivial topology receive a complex weight after the anomalous rotation. This changes the underlying physics and gives an inequivalent theory. Although not the topic of discussion here, note that this factor introduces a sign problem if one wishes to study this physics via Monte Carlo simulations. Here I have treated all N_f flavors equivalently; this corresponds to dividing the conventionally defined CP violation angle, to be discussed later, equally among the flavors, i.e. effectively $\phi = \Theta/N_f$.

The necessary involvement of both small and large eigenvalues warns of the implicit danger in attempts to separate infrared from ultraviolet effects. When the anomaly is concerned, going to short distances is not sufficient for ignoring non-perturbative effects related to topology.

4. A discrete chiral symmetry

I now return to the effective Lagrangian language of before. For the massless theory, the symmetry under $\sigma \leftrightarrow -\sigma$ indicates that we expect at

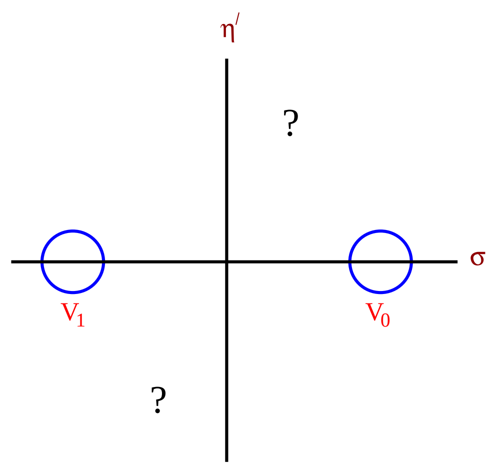

Figure 4: We have two minima in the σ, η' plane located at $\sigma = \pm v$ and $\eta' = 0$. Can we find any other minima?

least two minima for the effective potential considered in the σ, η' plane. These are located as sketched in Fig. 4. Do we know anything about the potential elsewhere in this plane? The answer is yes, indeed there are actually N_f equivalent minima.

As noted before, due to the anomaly the singlet rotation

$$\psi_L \rightarrow e^{i\phi}\psi_L \tag{11}$$

is not a valid symmetry of the theory for generic values of the angle ϕ. On the other hand, flavored chiral symmetries should survive, and in particular

$$\psi_L \rightarrow g_L\psi_L = e^{i\phi_\alpha\lambda_\alpha}\psi_L \tag{12}$$

should be a valid symmetry for any set of angles ϕ_α. The point of this section is that, for special special discrete values of the angles, the rotations in Eq. 11 and Eq. 12 can coincide. At such values the singlet rotation is a valid symmetry. In particular, note that

$$g = e^{2\pi i\phi/N_f} \in Z_{N_f} \subset SU(N_f). \tag{13}$$

Thus a valid discrete symmetry involving only σ and η' is

$$\begin{array}{rl} \sigma \rightarrow & \sigma\cos(2\pi/N_f) + \eta'\sin(2\pi/N_f) \\ \eta' \rightarrow & -\sigma\sin(2\pi/N_f) + \eta'\cos(2\pi/N_f). \end{array} \tag{14}$$

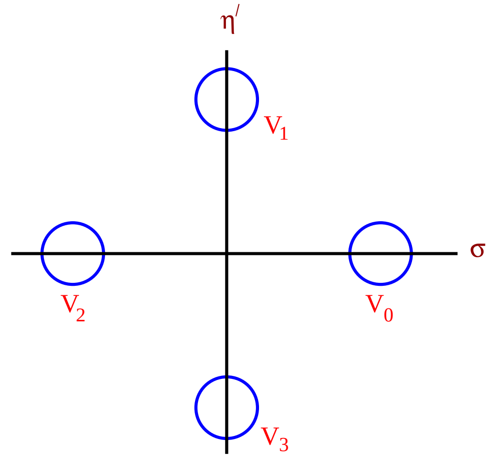

Figure 5: For four flavors we have four equivalent minima in the σ, η' plane. This generalizes to N_f minima with N_f flavors.

The potential $V(\sigma, \eta')$ has a Z_{N_f} symmetry manifested in N_f equivalent minima in the (σ, η') plane. For four flavors this structure is sketched in Fig. 5.

This discrete flavor singlet symmetry arises from the trivial fact that Z_N is a subgroup of both $SU(N)$ and $U(1)$. At the quark level the symmetry is easily understood since the 't Hooft vertex, responsible for the chiral anomaly, receives one factor from every flavor. With N_F flavors, these multiply together making

$$\psi_L \to e^{2\pi i/N_f} \psi_L \tag{15}$$

a valid symmetry even though rotations by smaller angles are not.

The role of the Z_N center of $SU(N)$ is illustrated graphically in Fig. 6, taken from Ref. [7]. Here I plot the real and the imaginary parts of the traces of 10,000 $SU(3)$ matrices drawn randomly with the invariant group measure. The region of support only touches the $U(1)$ circle at the elements of the center. All elements lie on or within the curve mapped out by elements of form $\exp(i\phi\lambda_8)$. Fig. 7 is a similar plot for the group $SU(4)$.

5. Massive quarks

As discussed earlier and illustrated in Fig. 5, a quark mass term $-m\overline{\psi}\psi \sim -m\sigma$ is represented by a "tilting" of the effective potential. This selects one of the minima in the σ, η' plane as the true vacuum. For masses small compared to the scale of QCD, the other minima will persist, although due to the flat

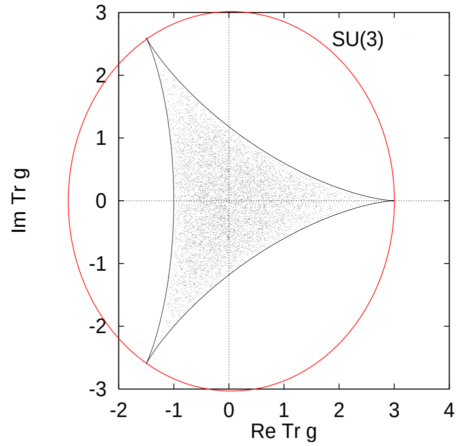

Figure 6: The real and imaginary parts for the traces of 10,000 randomly chosen $SU(3)$ matrices. All points lie within the boundary representing matrices of the form $\exp(i\phi\lambda_8)$. The tips of the three points represent the center of the group. The outer curve represents the boundary that would be found if the group was the full $U(1)$. Taken from Ref. [7].

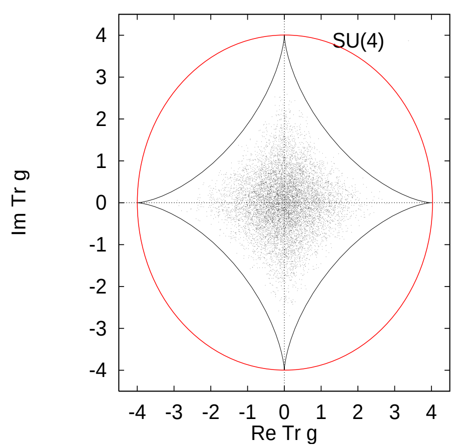

Figure 7: The generalization of Fig. 6 to $SU(4)$. The real and imaginary parts for the traces of 10,000 randomly chosen $SU(4)$ matrices. Taken from Ref. [7].

flavor non-singlet directions, some of them will become unstable under small fluctuations. Counting the minima sequentially with the true vacuum having $n = 0$, each is associated with small excitations in the pseudo-Goldstone directions having an effective mass of $m_\pi^2 \sim m\cos(2\pi n/N_f)$. Thus when N_f exceeds four, there will be more than one meta-stable state.

5.1. Twisted tilting

Conventionally the mass tilts the potential downward in the σ direction. However, it is interesting to consider tilts in other directions in the σ, η' plane. This can be accomplished by doing an anomalous rotation on the mass term

$$\begin{array}{rcl} -m\overline{\psi}\psi & \rightarrow & -m\cos(\phi)\overline{\psi}\psi - im\sin(\phi)\overline{\psi}\gamma_5\psi \\ & \sim & -m\cos(\phi)\sigma - m\sin(\phi)\eta' \end{array} \tag{16}$$

Were it not for the anomaly, this would just be a redefinition of fields. However the same effect that gives the η' its mass indicates that this new form for the mass term gives an inequivalent theory. As $i\overline{\psi}\gamma_5\psi$ is odd under CP, this theory is explicitly CP violating.

The conventional notation for this effect involves the angle $\Theta = N_f\phi$. Then the Z_{N_f} symmetry amounts to a 2π periodicity in Θ. As Fig. 8 indicates, at special values of the twisting angle ϕ, there will exist two degenerate minima. This occurs, for example, at $\phi = \pi/N_f$ or $\Theta = \pi$. As the twisting increases through this point, there will be a first order transition as the true vacuum jumps from one minimum to the next.

5.2. Odd N_f

One interesting consequence of this analysis is the behavior of QCD with an odd number of flavors. The group $SU(N_f)$ with odd N_f does not include the element -1. In particular, the Z_{N_f} structure is not symmetric under reflections about the η' axis. Fig. 9 sketches the situation for $SU(3)$. One immediate consequence is that positive and negative mass are not equivalent. Indeed, a negative mass corresponds to $\Theta = \pi$ where a spontaneous breaking of CP is expected. In this case the simple picture sketched in Fig. 1 no longer applies.

At $\Theta = \pi$ the theory lies on top of a first order phase transition line. A simple order parameter for this transition is the expectation value for the η' field. As this field is odd under CP symmetry, this is another way to see that negative mass QCD with an odd number of flavors spontaneously breaks CP.

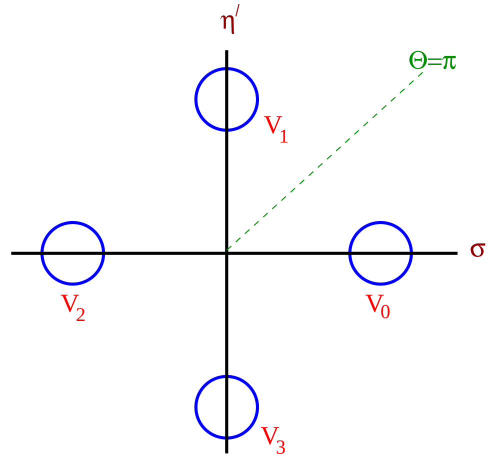

Figure 8: With massive quarks and a twisting angle of $\phi = \pi/N_f$, two of the minima in the σ, η' plane become degenerate. This corresponds to a first order transition at $\Theta = \pi$.

This does not contradict the Vafa-Witten theorem [17] because in this regime the fermion determinant is not positive definite.

Note that the asymmetry in the sign of the quark mass is not easily seen in perturbation theory. Any quark loop in a perturbative diagram can have the sign of the quark mass flipped by a γ_5 transformation. It is only through the subtleties of regulating the divergent triangle diagram [15], [16] that the sign of the mass enters.

A special case of an odd number of flavors is one-flavor QCD. In this case the anomaly removes all chiral symmetry and there is a unique minimum in the σ, η' plane, as sketched in Fig. 10. This minimum does not occur at the origin, being shifted to $\langle\overline{\psi}\psi\rangle > 0$ by 't Hooft vertex, which for one flavor is just an additive mass shift [18]. Unlike the case with more flavors, this expectation cannot be regarded as a spontaneous symmetry breaking since there is no chiral symmetry to break. Any regulator that preserves a remnant of chiral symmetry must inevitably fail [9]. Note also that there is no longer the necessity of a first order phase transition at $\Theta = \pi$. It has been argued [19] that for finite quark mass such a transition can occur if the mass is sufficiently negative, but the region around vanishing mass has no distinguishing structure.

One feature of one-flavor QCD is that the renormalization of the quark mass is not multiplicative when non-perturbative effects are taken into account. The additive mass shift is generally scheme dependent since the de-

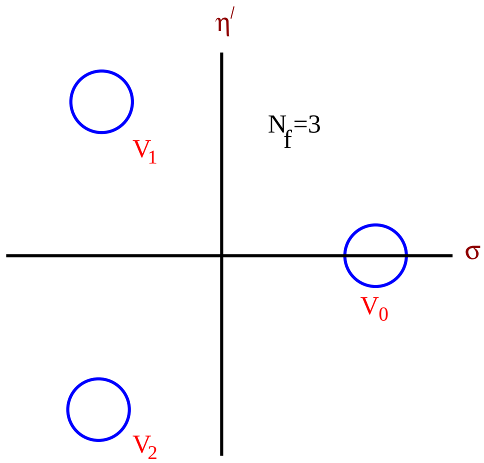

Figure 9: For odd N_f, such as the $SU(3)$ case sketched here, QCD is not symmetric under changing the sign of the quark mass. Negative mass corresponds to taking $\Theta = \pi$.

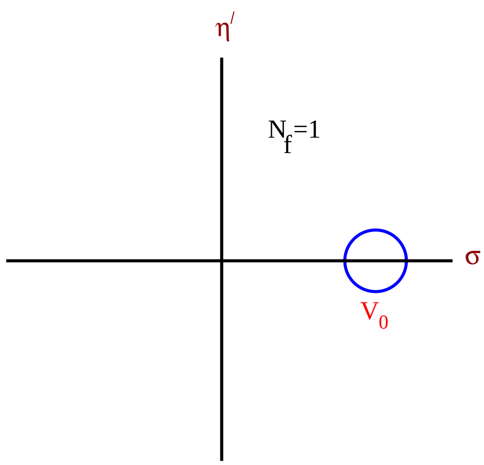

Figure 10: The effective potential for one-flavor QCD with small quark mass has a unique minimum in the σ, η' plane. The minimum is shifted from zero due to the effect of the 't Hooft vertex.

tails of the instanton effects depend on scale. This is the basic reason that a massless up quark is not a possible solution to the strong CP problem [8].

Because of this shift, the conventional variables Θ and m are singular coordinates for the one-flavor theory. A cleaner set of variables would be the coefficients of the two possible mass terms $\overline{\psi}\psi$ and $i\overline{\psi}\gamma_5\psi$ appearing in the Lagrangian. The ambiguity in the quark mass is tied to rough gauge configurations with ambiguous winding number. This applies even to the formally elegant overlap operator; when rough gauge fields are present, the existence of a zero mode can depend on the detailed operator chosen to project onto the overlap circle. Smoothness conditions imposed on the gauge fields to remove this ambiguity appear to conflict with fundamental principles, such as reflection positivity [20].

6. Varying N_f

The Z_{N_f} symmetry discussed here is a property of the fermion determinant and is independent of the gauge field dynamics. In Monte Carlo simulation language, this symmetry appears configuration by configuration. With N_f flavors, we always have $|D| = |e^{2\pi i/n_f} D|$ for any gauge field. This discrete chiral symmetry is inherently discontinuous in N_f. This non-continuity lies at the heart of the controversy over the rooted staggered quark approximation to lattice gauge theory. The details of this issue are extensively discussed elsewhere [9], but the essence is that the four species inherent with staggered quarks give rise to an unphysical extra Z_4 which current algorithms do not remove.

It is possible to interpolate between various numbers of flavors by adjusting the quark masses. Construct an $SU(N)$ valued effective field $\Sigma = \exp(i\pi_\alpha\lambda_\alpha)$. If, for instance, we give one flavor a large mass, this will drive one component of Σ to unity

$$\Sigma_{N_f} \longrightarrow \begin{pmatrix} \Sigma_{N_f-1} & 0 \\ 0 & 1 \end{pmatrix}, \tag{17}$$

leading one to the above discussion with one less flavor. A complication arises since the breaking of the $SU(N_f)$ flavor symmetry by a non-singlet mass term allows mixing of the η' field with the flavored analog of the η. As the heavier quark mass increases we should adjust what is meant by η', but qualitatively the transformation from, say, four to three flavors should look

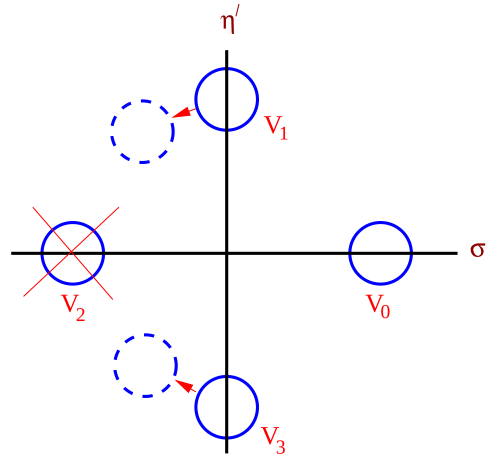

Figure 11: As one takes the four flavor theory and increases the mass of one quark, one of the four original minima of the effective potential should disappear while the others rearrange to give the final three fold symmetry.

something like what is sketched in Fig. 11, where one of the minima moves up to disappear and the others rearrange.

Continuing down from an odd number of flavors to an even number, then the two degenerate minima at negative mass should merge, as sketched in Fig. 12. For arbitrary masses the situation becomes increasingly complicated. In Ref. [21] the expected phase diagram is mapped out for the case of three flavors with arbitrary real masses. There it is shown that there are large regions both with and without a first order transition at $\Theta = \pi$.

The first order transition at $\Theta = \pi$ remains robust as long as multiple lightest quarks are degenerate and their masses remain in the regime where chiral expansions make sense. Regardless of any heavier quarks, the above symmetry arguments hold for the light quarks when the heavier masses are held fixed. Only when the lightest quark is non-degenerate can a gap appear separating the region of spontaneous CP violation at negative quark mass from zero mass. The size of this gap is controlled by the heavier quark masses.

7. Summary

I have discussed how the anomalous breaking of the classical $U(1)$ axial symmetry in QCD interplays with the spontaneously broken flavored axial symmetries. With N_f massless quarks a flavor singlet discrete Z_{N_f} chiral

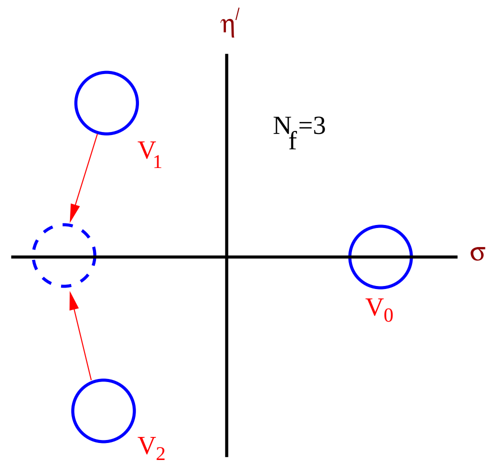

Figure 12: Going from three to two flavors by increasing the mass of the strange quark should result in two of the minima of the effective potential merging into one.

symmetry is left behind. This provides an intuitive interpretation of the strong CP violating angle Θ in terms of effective meson fields. As a consequence, a first order transition is generally expected at $\Theta = \pi$ and $m \neq 0$. This is quite robust and can only be avoided if one quark is considerably lighter than the others. At $\Theta = \pi$ the QCD Lagrangian is CP invariant; so, the transition represents a spontaneous breaking of this discrete symmetry. The physics of the anomaly indicates that the signs of quark masses can be significant, something that is not naturally interpreted perturbatively. In the special case of one flavor, all chiral symmetry is lost. One consequence is that the mass of a non-degenerate light quark is unprotected from an additive and scheme dependent renormalization. Furthermore, any proposed regulator that maintains an exact chiral symmetry for the one flavor case must fail.

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