Gauge field topology and the hadron spectrum

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Abstract. Topologically non-trivial gauge field configurations are an interesting aspect of non-abelian gauge theories. These become particularly important upon quantizing the theory, especially through their effect on the pseudo-scalar spectrum. These effects are closely tied to chiral anomalies and the possibility of CP violation in the strong interactions.

1. INTRODUCTION

It has long been recoginized that topology plays a fascinating role in the theory of gauge fields. This appears already at the level the classical theory, with topology at the heart of a multitude of nontrivial exact solutions to the Yang-Mills field equations. The quantum theory inherits many interesting features through topological excitations appearing in the path integral. Topology is also intimately connected with the behavior of fermion fields through the index theorem. This directly leads to the the well known anomalies in the chiral symmetries of the theory, which are crucial to understnding certain aspects of the spectrum of the pseudoscalar mesons.

This presentation is meant to be an elementary overview of how this physics fits in with our understanding of QCD. Much of this material is adapted from the more extensive review in Ref. [1]. I begin in section 2 with a brief discussion of how topology becomes relevant to classical gauge theories through boundary conditions. Section 3 goes on to discuss some issues that arise in the quantum theory and their implications for the continuum limit of the lattice theory. I then turn in section 4 to how topology becomes particularly crucial through its effect on quark fields via the index theorem. This is important to understanding chiral symmetry and the pseudo-scalar spectrum, as will be discussed in sections 5 and 6. Section 7 explores some unresolved issues that arise on implementing these ideas with a lattice cutoff. Section 8 provides a brief recap of the earlier sections.

2. CLASSICAL GAUGE FIELDS

One path towards understanding how topology enters a gauge theory is through boundary conditions. It is natural to impose that the gauge field tensor $F_{\mu\nu}$ vanishes at spatial infinity. This alone, however, is not sufficient to say that the vector potential A_μ also vanishes there. Indeed the only implication is that the potential goes to the form of a pure gauge; i.e. as $|x_\mu| \to \infty$ this requires

$$A_\mu \to -ih^\dagger \partial_\mu h \tag{1}$$

where $h(x)$ is an element of the gauge group.

Now formally spatial infinity is a sphere S_3 and the gauge group can also contain spheres. For example, a general $SU(2)$ element can be written in the form $h = a_0 + \vec{a} \cdot \vec{\sigma}$ where $a_\mu^2 = 1$. That is, the group itself is an S_3. Higher groups generally contain multiple $SU(2)$ subgroups, to which the following discussion can be applied. The basic issue is that the gauge function $h(x)$ can wrap non-trivially around the group as x surrounds spatial infinity. Such mappings cannot be smoothly deformed onto each other. The space of all gauge fields where $F_{\mu\nu}$ vanishes at infinity then divides into topological "sectors" depending on how many times $h(x)$ at infinity wraps around the group.

Considering all gauge configurations of a given topology, one can search for one that minimizes the action. Being a local minimum, this is also automatically a solution to the Yang-Mills equations of motion. For a single wrapping, this is the famous "instanton" solution. Locating the action peak at the origin, this takes the form

$$A_\mu = \frac{-ix^2}{g(x^2 + \rho^2)} h^\dagger \partial_\mu h \tag{2}$$

where we take

$$h(x_\mu) = \frac{t + i\vec{\tau} \cdot \vec{x}}{\sqrt{\vec{x}^2 + t^2}} \in SU(2). \tag{3}$$

Note how the factor $\frac{x^2}{x^2+\rho^2}$ mollifies the singularity in $h^\dagger \partial_\mu h$ at the origin.

This solution depends on an arbitrary parameter ρ which characterizes the instanton size. Indeed, there must be such a scale parameter since the pure classical gauge theory is scale invariant. Integrating over space gives the classical instanton action

$$S_I = \frac{1}{2} \int d^4x \, \text{Tr} \, F_{\mu\nu} F_{\mu\nu} = \frac{8\pi^2}{g^2}. \tag{4}$$

Because of the inverse coupling dependence of this action, any physics associated with non-trivial topology is necessarily non-perturbative.

3. QUANTUM ISSUES

Of course for particle physics one is interested in the quantum theory. The usual procedure for this is to introduce a path integral involving a sum over all possible configurations of the gauge fields. There are several subtle and non-intuitive issues that arise upon quantization. In general the winding number

$$\nu = \frac{g^2}{16\pi^2} \int d^4x \, \text{Tr} F\tilde{F} \tag{5}$$

is robust under smooth field deformations. However in path integrals it is well known that typical paths are non-differentiable. On such it is not clear if the concept of topology is well defined. In the next subsections I discuss some of the subtleties that arise.

3.1. Positivity and the topological susceptibility

A natural quantity to study in the quantum theory is the topological susceptibility, defined as

$$\xi = \langle \frac{\nu^2}{V} \rangle = \left(\frac{g^2}{16\pi^2} \right)^2 \int d^4x \langle F\tilde{F}(x) \, F\tilde{F}(0) \rangle \tag{6}$$

This directly measures the typical quantum fluctuations in the winding number, and since it is the expectation of a square, it must be positive.[1] But the combination $F\tilde{F}$ as an operator is odd under time reversal. By reflection positivity [3], this means that for any non-zero separation x one must have

$$\langle F\tilde{F}(x) \, F\tilde{F}(0) \rangle < 0 \tag{7}$$

Thus the integral in Eq. (6) receives a negative contribution from all $x \neq 0$. In order to obtain a positive susceptibility, there must be a positive "contact" term from a singularity at $x = 0$. This observation emphasizes that long and short distance phenomena are intimately entwined when non-perturbative effects from topology are important. It is inherently dangerous to try to separate out perturbative effects by discussing only short distances. This point will arise again when I discuss the theory with quarks present.

3.2. The interplay of topology and asymptotic freedom

The well known phenomenon of asymptotic freedom [4, 5, 6, 7, 8] implies that the effective gauge coupling g depends on scale and decreases at short distances. On quantization the conformal invariance of the classical theory is

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[1] This is in the absence of fermions. With such present, a negative fermion determinant can give rise to a susceptibility that is formally negative [2].

lost and a scale enters the problem through what is called "dimensional transmutation" [9]. For the lattice theory this means that the bare coupling at the lattice scale, $g(a)$, must be taken to zero for the continuum limit. In effect, the lattice spacing a represents an ultraviolet cutoff on the theory. This decrease in the coupling is logarithmic

$$g^2(a) \sim \frac{1}{2\beta_0 \log(1/\Lambda a)} \tag{8}$$

where $\Lambda = \Lambda_{qcd}$ is an integration constant from the renormalization group equation. This is a dimensional parameter that sets the scale for particle masses in the quantum theory. More precisely, Λ satisfies

$$\Lambda_{qcd} = \frac{1}{a} e^{1/2\beta_0 g^2} g^{-\beta_1/\beta_0^2} (1 + O(g^2)) \qquad \longrightarrow_{a \to 0} \qquad \text{constant} \tag{9}$$

where

$$\beta_0 = \frac{1}{16\pi^2}(11 - 2N_f/3), \qquad \beta_1 = \left(\frac{1}{16\pi^2}\right)^2 (102 - 22N_f/3). \tag{10}$$

Here N_f denotes the number of quark species.

Because the coupling goes to zero, small classical instantons are suppressed in the continuum limit. Naively combining the classical action in Eq. (4) with the asymptotic freedom result gives

$$e^{-S_I} \sim e^{-8\pi^2/g(a)^2} \sim e^{-16\pi^2 \beta_0 \log(a\Lambda)} \sim a^{11-2N_f/3}. \tag{11}$$

This represents a strong suppression of instantons by a power law in the lattice spacing. For 3 flavors, this is by a^9 per unit lattice volume, or a^5 per unit physical volume. Because of this it is sometimes argued that one can ignore instantons at small lattice spacing.

This is a wrong conclusion. As will be discussed later, topological excitations directly affect the hadron spectrum in the continuum limit. In other words, they contribute to physics at order a^0. This difference arises because the above semi-classical argument ignores quantum fluctuations. Typical quantum paths are far from smooth, indeed, they are non-differentiable. The most direct way to see how topology modifies the hadron spectrum is through their influence on the quark fields, to which I now turn.

4. FERMIONS AND THE INDEX THEOREM

The interactions of the quarks with the gluon fields enter through the Dirac contribution to the action $S_f = \overline{\psi}(D+m)\psi$. In practice the kinetic part of the Dirac operator D satisfies gamma-five hermiticity,

$$D^\dagger = -D = \gamma_5 D \gamma_5. \tag{12}$$

In continuum discussions D is also anti-Hermitian and anti-commutes with γ_5. This latter condition is usually modified on the lattice, but that won't concern us for the moment.

A particularly important consequence of topology is the index theorem. This says that when the gauge field has non-trivial winding, the Dirac operator D will have exact zero modes, i.e. there exist functions that satisfy

$$D|\psi\rangle = 0. \tag{13}$$

Furthermore, on the space of zero modes γ_5 can be diagonalized. Thus these modes can be considered to be chiral

$$\gamma_5|\psi\rangle = \pm|\psi\rangle. \tag{14}$$

The theorem states that the winding number equals the difference between the number of right and left handed modes,

$$\nu = n_+ - n_-. \tag{15}$$

The index theorem implies that on a fixed gauge field configuration $\text{Tr}\gamma_5 = \nu$. At first sight this seems to be a strange result since, when thought of as a four by four matrix, $\text{Tr } \gamma_5 = 0$. But, as argued by Fujikawa, [10] this naive

conclusion must be modified in a regulated theory. To see this, it is natural to use the eigenstates of D to define the trace. Considering a complete set of eigenstates

$$D|\psi_i\rangle = \lambda_i |\psi_i\rangle, \tag{16}$$

a natural definition for the trace is

$$\text{Tr}\gamma_5 = \sum_i \langle \psi_i | \gamma_5 | \psi_i \rangle \tag{17}$$

All non-zero eigenstates occur in chiral pairs; for an eigenstate $|\psi_i\rangle$, then

$$D\gamma_5|\psi\rangle = -\lambda\gamma_5|\psi\rangle = \lambda^*\gamma_5|\psi\rangle \tag{18}$$

From this $|\psi\rangle$ and $|\gamma_5\psi\rangle$ are orthogonal when $\lambda \neq 0$, and in turn the space spanned by $|\psi\rangle$ and $|\gamma_5\psi\rangle$ gives no contribution to $\text{Tr}\gamma_5$. Only the zero modes count towards the trace and one has the basic result

$$\text{Tr}\gamma_5 = \sum_i \langle \psi_i | \gamma_5 | \psi_i \rangle = \nu. \tag{19}$$

Since γ_5 arises from a traceless four by four matrix, it is natural to ask what happened to the opposite chirality states? In a continuum discussion it is easiest to think of them as being lost at "infinity," i.e. they are at an energy beyond the cutoff. On the lattice there is no real infinity, and the answer depends on the details of the fermionic action. With Wilson fermions [11], the would-be zero modes can acquire a small real part and compensating real eigenvalues appear in the doubler region. Including all states, γ_5 remains traceless. With overlap fermions [12, 13], all eigenvalues of the Dirac operator lie on a circle; for every zero mode there is a compensating real eigenvalue with opposite chirality on the far side of the circle.

This brings us back to the earlier conclusion that this phenomena responsible for the anomaly involves both long and short distances. Even for large instantons, there is a compensating mode which is at or beyond the cutoff region. And for small instantons, they can combine with these other states and "fall through the lattice." The details depend on the specific cutoff in place; indeed, these can be scheme and scale dependent.

The way the zero modes from topology bring about the anomaly can be nicely understood in terms of the fermionic measure [10]. In particular, the fact that γ_5 is formally not traceless means that the change of variables

$$\psi \rightarrow e^{i\gamma_5\theta}\psi \tag{20}$$

changes the fermion measure in the path integral

$$(d\psi d\overline{\psi}) \rightarrow e^{i\theta \text{Tr}\gamma_5}(d\psi d\overline{\psi}) = e^{i\nu\theta}(d\psi d\overline{\psi}). \tag{21}$$

Thus such a change in variables inserts a factor of $e^{i\nu\theta}$ into the path integral weight. This gives rise to an inequivalent theory. Starting with the naive path integral, this rotation gives what is often called the "Theta vacuum," the ground state of an independent and physically distinct theory in which CP symmetry is explicitly broken.

4.1. Fixed topology

Because the space of smooth fields breaks up into distinct sectors, it turns out that in a simulation tunneling between these sectors is difficult. This gives rise to long correlation times. This issue becomes even more severe when the quarks become light. In this case the the near zero modes suppress configurations of non-trivial topology.

This raises the question of what would happen if one were to ignore this tunneling and work in a sector of fixed total topology. This can be implemented formally by integrating over Θ to select out the sector of interest. Thus consider the fixed topology path integral

$$Z_\nu(m_q) = \int \frac{d\Theta}{2\pi} e^{i\nu\Theta} Z(m_q, \Theta) \tag{22}$$

In some ways this seems like a rather perverse thing to do since each Theta vacuum represents a physically different theory. Nevertheless it has been argued [14] that as the system volume becomes large one can still obtain valid physics.

As the volume increases the path integral grows/decreases exponentially in the free energy of the four dimensional statistical system being simulated

$$Z(m_q, \Theta) = \int (dA)(d\overline{\psi})(d\psi) \, e^{-S(m_q, \Theta)} \; = \; e^{-VF(m_q, \Theta)} \tag{23}$$

Although each value of (m_q, Θ) represents an physically different field theory, as $V \to \infty$ with fixed ν the integral over Θ in Eq. (22) will be dominated by the saddle point at $\Theta = 0$. In physical terms, at large enough volume a few instantons can "hide behind the moon." The conclusion is that at large volume, despite the long correlation time in the total winding number, local observables are presumably much better behaved.

5. THE PSEUDO-SCALAR SPECTRUM

Possibly the most direct physical consequences of topology appear in the spectrum of the pseudo-scalar mesons. Consider two flavor QCD with light but non-degenerate quark masses. As usual, label the quark fields as u and d. There are four natural pseudo-scalar bilinears in the quark fields

$$\begin{aligned} &\overline{iu}\gamma_5 u \\ &\overline{id}\gamma_5 d \\ &\overline{iu}\gamma_5 d \sim \pi_+ \\ &\overline{id}\gamma_5 u \sim \pi_- \end{aligned} \tag{24}$$

All of these combinations involve a helicity flip, for example $\overline{u}\gamma_5 u = \overline{u}_L \gamma_5 u_R + \overline{u}_R \gamma_5 u_L$. Now a well known property of gauge theories is the suppression of helicity flip processes in the chiral limit. This naively suggests that the mixing of $\overline{u}\gamma_5 u$ with $\overline{id}\gamma_5 d$ should be suppressed by a factor of $m_u m_d$. Without such mixing there should be two light neutral pseudo-scalar pions, one primarily made of up quarks and the second from down quarks.

This is of course wrong. There is only one neutral pion, not two. It is the anomaly that strongly couples the up and down combinations through what is commonly called the effective "t'Hooft vertex" [15]. In this way the symmetric combination

$$\eta' \sim \overline{u}\gamma_5 u + \overline{d}\gamma_5 d \tag{25}$$

is not a pseudo-Goldstone boson and acquires a mass of order the QCD scale

$$M_{\eta'} \propto \Lambda_{qcd} + O(m_u, m_d). \tag{26}$$

The mixing responsible for the eta prime mass leaves behind the orthogonal combination

$$\pi_0 \sim \overline{iu}\gamma_5 u - \overline{id}\gamma_5 d, \tag{27}$$

which is, of course, the neutral pion. In the process, isospin breaking is suppressed to a higher order in the chiral expansion

$$M_{\pi_0}^2 = M_{\pi_\pm}^2 - O((m_u - m_d)^2). \tag{28}$$

The basic result is that the η' meson is not a Goldstone boson because

$$\psi \to e^{i\gamma_5 \theta} \psi \tag{29}$$

is not a symmetry of the quantum theory. In the absence of such a symmetry, the mass of the η' is proportional to Λ_{qcd}, the scale of the strong interactions, and does not vanish as the quark masses go to zero

$$M_{\eta'}^2 \propto \frac{1}{a^2} e^{1/\beta_0 g^2} g^{-\beta_1/\beta_0^2} (1 + O(g^2)) \sim a^0. \tag{30}$$

This behavior was alluded to earlier and is a direct indication of how the semi-classical estimate from Eq. (11) substantially understates the importance of topology in the quantum theory.
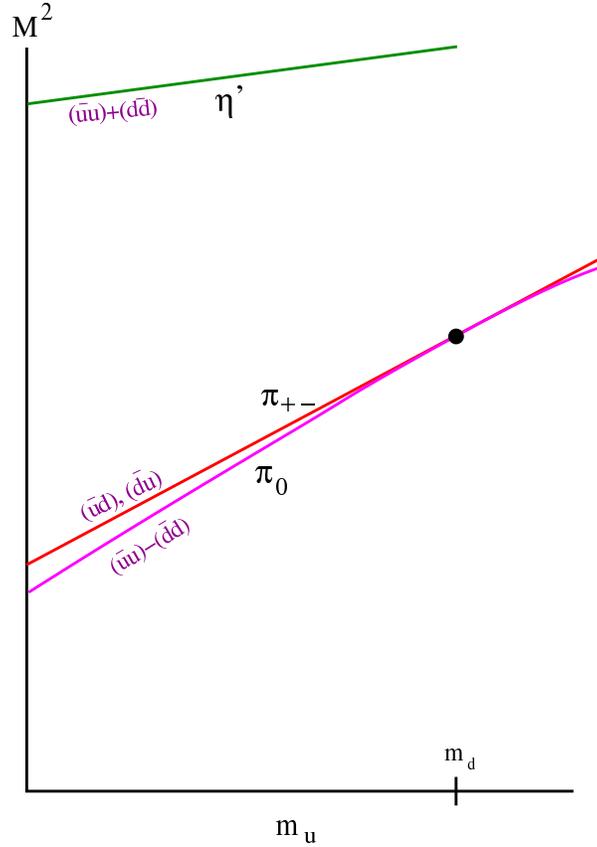
FIGURE 1. A mass gap will persist when the up quark mass vanishes but the down quark remains massive. Higher order effects split the charged and neutral pions.

6. ISOSPIN BREAKING AND QUARK MASSES

I now turn to some interesting properties of the theory as a function of the up quark mass m_u when the down quark mass m_d is fixed at a non-zero value. Consider the situation where both are light compared to the strong scale, Then chiral symmetry predicts

$$M_\pi^2 \propto \frac{m_u + m_d}{2} + O(m_q^2). \tag{31}$$

As discussed above, the eta prime remains massive with $M_{\eta'} \sim \Lambda_{qcd}$. But an important observation is that a finite mass gap remains if only the up quark is massless. This is sketched in Fig. 1.

The effect of isospin breaking on the pion masses is of higher order in the quark masses[2]

$$M_{\pi_\pm}^2 - M_{\pi_0}^2 \propto (m_d - m_u)^2, \tag{32}$$

At this quadratic order in the quark mass difference, isospin breaking is expected to induce some mixing between the neutral pion, the eta prime, and pseudo-scalar glueballs. This mixing is expected to make the neutral pion become the lightest of the three pions.

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[2] I ignore the splitting coming from electromagnetism, which dominates the physical mass difference. In the following I explore the hypothetical situation where the mass splitting is large and dominates the electromagnetic part.
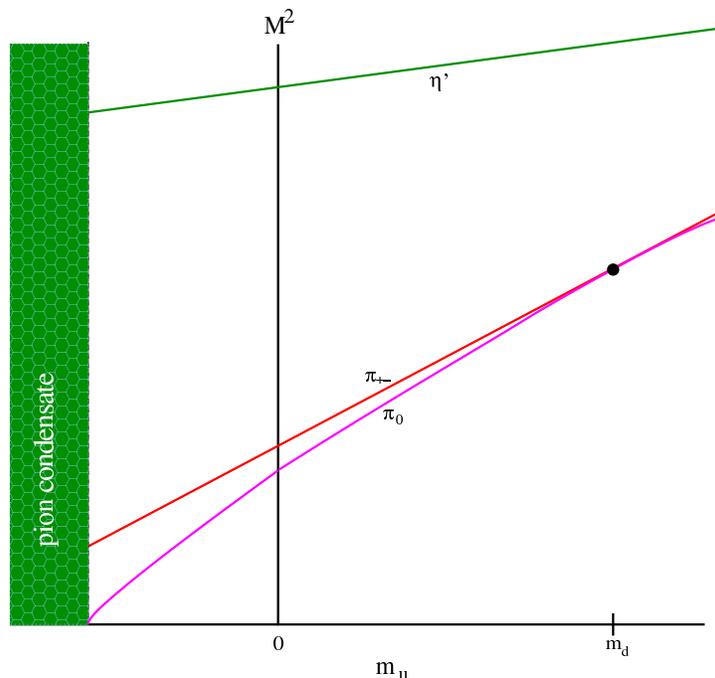
FIGURE 2. No singularity is expected as the up quark mass passes through zero at fixed down quark mass. On the other hand, for sufficiently negative up quark mass the neutral pion can condense into a CP violating phase.

6.1. The Dashen phase

Since there is a mass gap at vanishing up quark mass, it is natural to ask what would happen if the up quark mass were to become negative. In effective chiral Lagrangian models, both linear and nonlinear [1, 16, 17, 18], the spectrum remains well behaved, without any singularity on passing through the zero mass point. There is, however, a natural limit to the size of the mass splitting when the mixing with the eta prime is large enough to make the neutral pion massless. Beyond that point the neutral pion field acquires an expectation value and one finds a new phase in which CP is spontaneously broken. Because the product of the quark masses is negative, this occurs in a region where the parameter Θ formally takes the value π. The possibility of such a behavior was conjectured some time ago by Dashen [19]. This behavior is qualitatively sketched in Fig. 2.

In the effective Lagrangian models this is an Ising like transition occurring at a negative up-quark mass. The order parameter is the expectation value of the neutral pion, $\langle \pi_0 \rangle \neq 0$. Because the pion is CP odd, this symmetry is spontaneously broken in the Dashen phase. The qualitative phase diagram as a function of the two quark masses is sketched in Fig. 3. This structure gives rise to several remarkable conclusions. First, a mass gap persists at $m_u = 0$ when $m_d \neq 0$. This means that despite the fact that the Dirac operator can have small eigenvalues, all long distance physics is exponentially suppressed by this mass gap. Second, the presence of a second order transition when neither m_u nor m_d vanishes shows that it is possible to have a divergent correlation length and long distance physics in a regime where the Dirac operator has no small eigenvalues.

Fig. 3 also illustrates important symmetries. Note the reflection symmetry about the 45 degree $m_u + m_d$ direction, and another about the $m_u - m_d$ direction. These symmetries correspond to an isoscalar mass term $m_u + m_d$ and an isovector mass $m_u - m_d$. Each of these are separately protected from any additive renormalization. However there is no symmetry between these possible mass terms. Because of this, the up and down quark masses are not individually protected. In particular non-perturbative mass renormalization does not maintain the perturbative property of being "flavor blind."

This lack of symmetry in the individual quark masses arises because non-perturbative contributions can mix quark masses. Fig. 4 shows one way to think of this mixing. The eta prime and neutral pion mesons are both non-trivial mixtures of $\overline{u}u$ and $\overline{d}d$ quark-antiquark pairs. Because these mesons are non-degenerate, their contribution to the mixing of $\overline{u}u$ and $\overline{d}d$ cannot cancel. Thus a small down quark mass will induce an effective up quark mass, even if the
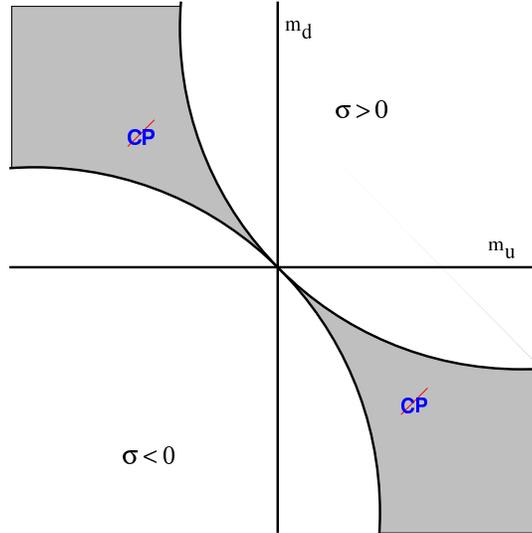
FIGURE 3. The qualitative phase diagram for two flavor QCD as a function of the up and down quark masses. The CP violating Dashen phase occurs in the region where the two masses have opposite sign.
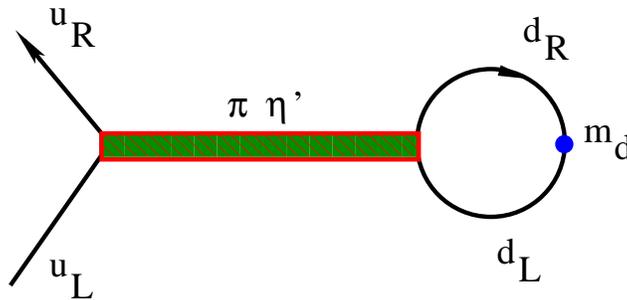

FIGURE 4. A small down quark mass can induce an effective up quark mass through intermediate meson exchange, even if the up quark is perturbatively massless.

up quark is massless in perturbation theory. This induced up quark mass is of form

$$\delta m_u \propto \frac{(M_{\eta'} - M_\pi)}{\Lambda_{qcd}} m_d. \tag{33}$$

For three flavors there is also a contribution from the strange quark giving $\delta m_u \propto \frac{m_d m_s}{m_d + m_s}$. Note that this is the same form as the Kaplan and Manohar ambiguity in chiral perturbation theory [20].

This discussion shows that the concept of a single massless quark is not renormalization group invariant. Indeed, is there any experimental way to tell if $m_u = 0$? It is often stated that the strong CP problem would be solved if the up quark mass vanishes. But can this make any sense if such a concept is ill-defined? This cannot be answered in the context of the $\overline{MS}$ scheme since that is perturbative and these effects are purely non-perturbative.

7. THE LATTICE

So, a non-perturbative approach is necessary to understand the quark masses. This leads us directly to the lattice. Naively all one needs to do is adjust the lattice parameters to get the hadron spectrum, and then read off the quark masses to answer such questions as whether $m_u = 0$.

But this leaves many open questions. There are many different lattice formulations. Are the quark masses unique between them? What defines the quark mass anyway? One could look for poles in the quark propagator, but the
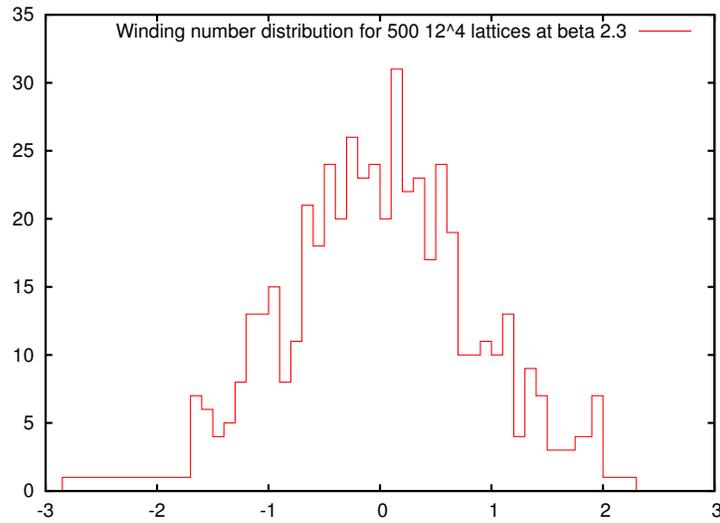
FIGURE 5. The distribution of topological charge on an ensemble of SU(2) gauge theory lattices as constructed from a set of loops around the lattice hyper-cubes [21]. Without some sort of smoothing the distribution does not show peaks at integer values.
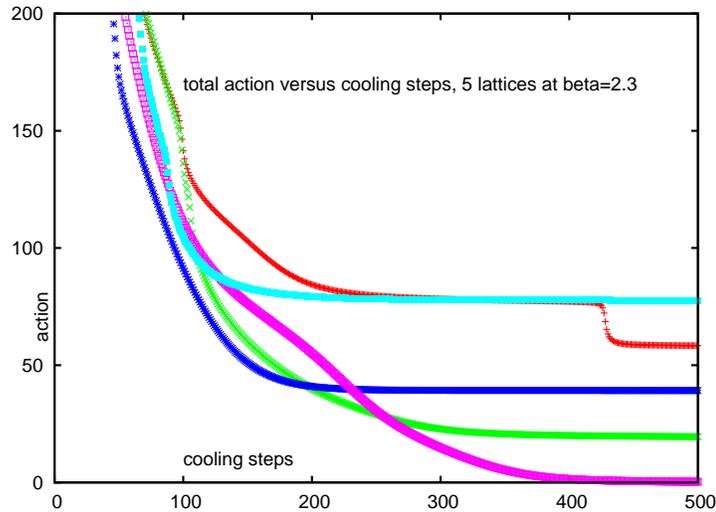

FIGURE 6. The lattice action for five SU(2) lattices as cooling removes dislocations. The action settles to multiples of the classical instanton value.

propagator is gauge dependent and the result might depend on the gauge chosen. These are all non-trivial questions that have not been fully answered.

The quark mass is closely connected with topology; if $m_u = 0$ the topological susceptibility should vanish. But how does one define topology on a discrete lattice? Small instantons can "fall through the lattice" and the concept of topology is lost at the outset. One can construct a combination of loops around a hyper-cube that reduces to $F\tilde{F}$ in the naive continuum limit. There are a variety of ways to do this, but the resulting topological charge is not generally an integer. One example is shown in Fig. 5, taken from [21].

The topological charge of individual configurations can be driven to integers by various cooling algorithms. These remove rough configurations and the action settles into multiples of the classical instanton result. The result of such a cooling process is shown in Fig. 6, taken from Ref. [21]. Over the years there have been many such studies with a variety of methods [22, 23, 24, 25, 26].
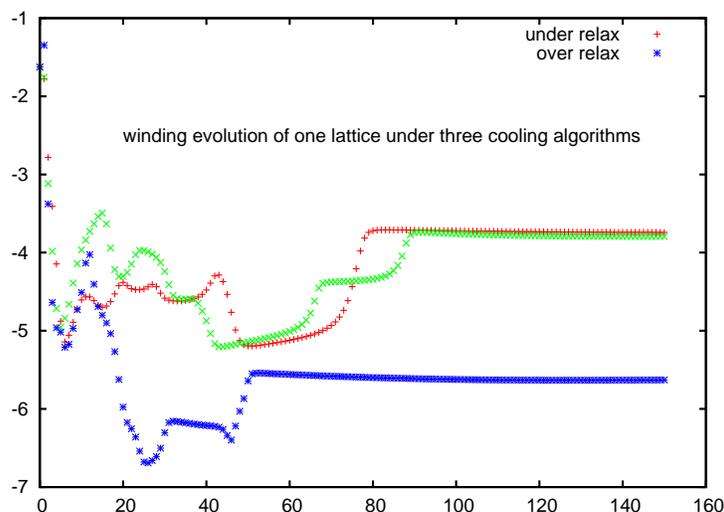
FIGURE 7. The evolution of the action for a single lattice under different cooling algorithms. The result can depend on the details of the algorithm as well as the time of cooling.

Cooling often gives a stable result, but ambiguous cases do appear. Indeed, the final winding can depend on details of the cooling algorithm. For example, Fig. 7, also taken from Ref. [21], shows the evolution of a single configuration under several rather different cooling algorithms. This leaves a variety of open questions. With which gauge action should one cool, particularly when dynamical fermions are present? How long should one cool? With too much cooling, will small "instantons" eventually collapse?

At this point I remark on a rather technical point. If one puts a bound on how far any individual plaquette can be from the identity, then it is possible to implement a unique interpolation of the gauge fields through the lattice hyper-cubes. If this "admissibility condition" is imposed, the winding number becomes well defined. Specifically, Ref. [27] has shown that if the trace if each plaquette in an SU(3) gauge theory takes a value $P <\sim .03$, then instantons can no longer collapse and the configuration has a unique winding number.

The problem with the admissibility constraint is that it requires a non-Hermitian Hamiltonian. The transfer matrix relates the path integral to the Hamiltonian through the transfer matrix

$$Z = \text{Tr} e^{-\beta H} = \text{Tr}(e^{-aH})^{N_t}. \tag{34}$$

A Hermitian H requires $\langle \psi | e^{-aH} | \psi \rangle > 0$ for every state ψ. It can be shown [28] that this requires the plaquette weight to be analytic over the gauge group. That in itself is inconsistent with the admissibility constraint.

Can the index theorem provide another approach to defining the topological charge? For example, count the small real eigenvalues of the Wilson operator and use the result as a definition of the charge? This becomes tricky since at finite cutoff these are not exact zeros but are spread over a region of the real axis. Thus the word "small" is a bit arbitrary. The unresolved question is whether the eigenvalue distribution in the first Wilson "circle" goes to zero fast enough to remove this ambiguity.

One might consider the zero modes of the overlap operator, which are indeed exact zero modes. The problem is that the overlap operator is not unique, and depends on the chosen "domain wall height." The uniqueness of this again relies on the density of eigenvalues in the first Wilson "circle" going to zero sufficiently rapidly.

7.1. Should we care?

Should we care if there is a small ambiguity in defining topology? After all, this is not something directly measured in laboratory experiments. It is perhaps better to concentrated on something like the mass of the eta prime, which is clearly physical. Of course there is the Witten-Veneziano formula [29, 30] relating this to topology, but that is valid only in the large N_c limit while $N_c = 3$ for physics.

As discussed above, topology is closely related to the issue of whether $m_u = 0$ or not. Fig. 3 shows that there is no symmetry around the m_d axis. Is there perhaps some Ward identity that fixes the location of this line? Any such relation involves anomalous currents and thus must bring in the topological susceptibility. Thus any ambiguities in defining a vanishing quark mass or the topological susceptibility are directly coupled. And one should remember from [20] that these ideas are already ambiguous at the level of chiral perturbation theory.

8. SUMMARY

I hope I have convinced you that the role of topology in gauge theories is a rich and fascinating topic. There are important consequences for understanding the light hadron spectrum, and thus the topic is highly relevant to this meeting. It is also important to remember that conventional perturbation theory misses many of these issues, for example the mass mixing effects between species.

I have also discussed the fascinating phase structure that can appear with negative mass quarks. This includes a phase with pion condensation and spontaneous CP violation. While this is not directly relevant for the usual strong interactions, it might be interesting to consider such a mechanism to introduce CP violation in models of unification through a new strong dynamics.

ACKNOWLEDGMENTS

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