Local chiral fermions

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Summary

- A strictly local fermion action $\mathcal{D}(A)$
  - with one exact chiral symmetry $\gamma_5 \mathcal{D} = -\mathcal{D}\gamma_5$
  - describing two flavors; minimum required for chiral symmetry
  - a linear combination of two "naive" fermion actions (Borici)
- Space-time symmetries
  - translations plus 48 element subgroup of hypercubic rotations
  - includes odd parity transformations
  - renormalization can induce anisotropy at finite a

Chiral symmetry crucial to our understanding of hadronic physics

- pions are waves on a background quark condensate $\langle\overline{\psi}\psi\rangle$
- chiral extrapolations essential to practical lattice calculations

Anomaly removes classical $U(1)$ chiral symmetry

- $SU(N_f) \times SU(N_f) \times U_B(1)$
- non trivial symmetry requires $N_f \geq 2$

Minimally doubled chiral fermion actions have just 2 species

- Karsten 1981
- Wilczek 1987
- recent revival: MC, Borici, Bedaque Buchoff Tiburzi Walker-Loud

Motivations

- failure of rooting for staggered
- lack of chiral symmetry for Wilson
- computational demands of overlap, domain-wall approaches

Here I follow Borici's construction

- linear combination of two equivalent naive fermion actions

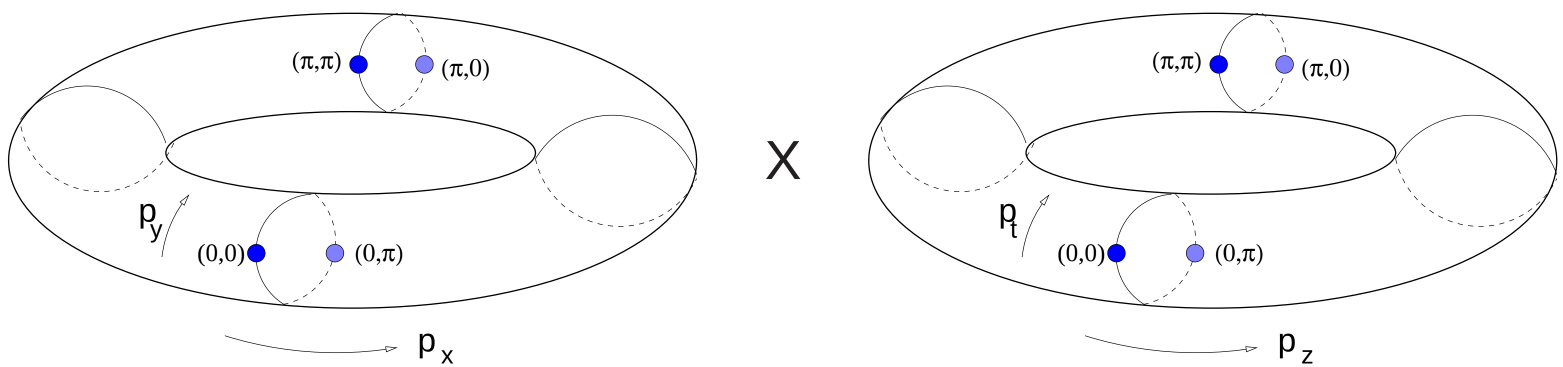
Start with naive fermions

- forward hop between sites $\gamma_\mu U$ unit hopping parameter for convenience
- backward hop between sites $-\gamma_\mu U^\dagger$
  - μ is the direction of the hop
  - U is the usual gauge field matrix
- 16 doublers
- Dirac operator D anticommutes with γ_5
  - an exact chiral symmetry
  - part of an exact $SU(4) \times SU(4)$ chiral algebra Karsten and Smit

In the free limit, solution in momentum space

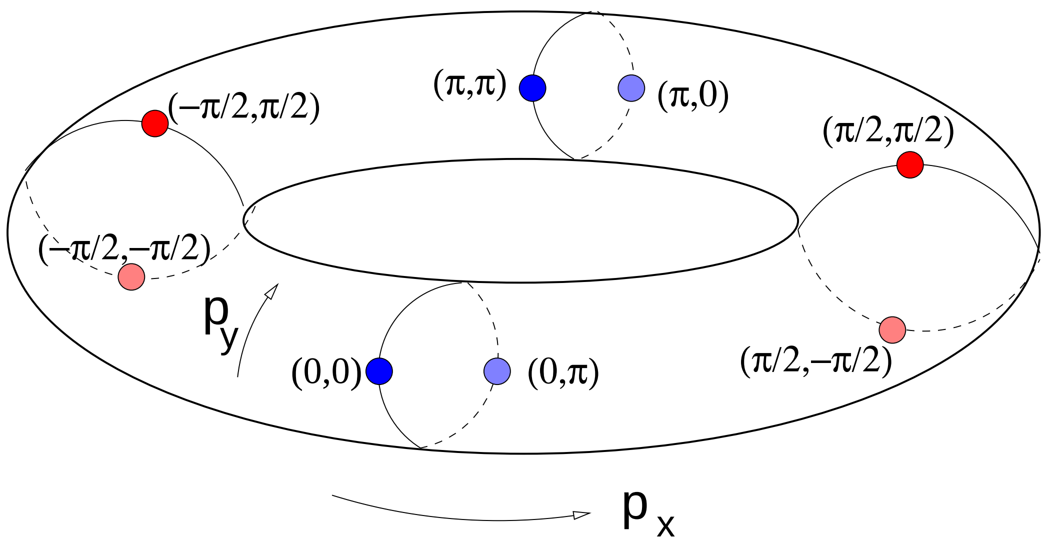
$$D(p) = 2i \sum_\mu \gamma_\mu \sin(p_\mu)$$

- for small momenta reduces to Dirac equation
- 15 extra Dirac equations for components of momenta near 0 or π



16 ‘‘Fermi points’’

Consider momenta maximally distant from the zeros: $p_\mu = \pm\pi/2$



Select one of these points, i.e. $p_\mu = +\pi/2$ for every μ

- $D(p_\mu = \pi/2) = 2i \sum_\mu \gamma_\mu \equiv 4i\Gamma$
- $\Gamma \equiv \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$
  - unitary, Hermitean, traceless 4 by 4 matrix

Now consider a unitary transformation

- $\psi'(x) = e^{-i\pi(x_1+x_2+x_3+x_4)/2}\,\Gamma\,\psi(x)$
- $\overline{\psi}'(x) = e^{i\pi(x_1+x_2+x_3+x_4)/2}\,\overline{\psi}(x)\,\Gamma$

- phases move Fermi points from $p_\mu \in \{0,\pi\}$ to $p_\mu \in \{\pm\pi/2\}$
- ψ' uses new gamma matrices $\gamma'_\mu = \Gamma\gamma_\mu\Gamma$
  - $\Gamma = \frac{1}{2}(\gamma_1+\gamma_2+\gamma_3+\gamma_4) = \Gamma'$
- new free action: $\overline{D}(p) = 2i\sum_\mu \gamma'_\mu \sin(\pi/2 - p_\mu)$

D and $\overline{D}$ physically equivalent

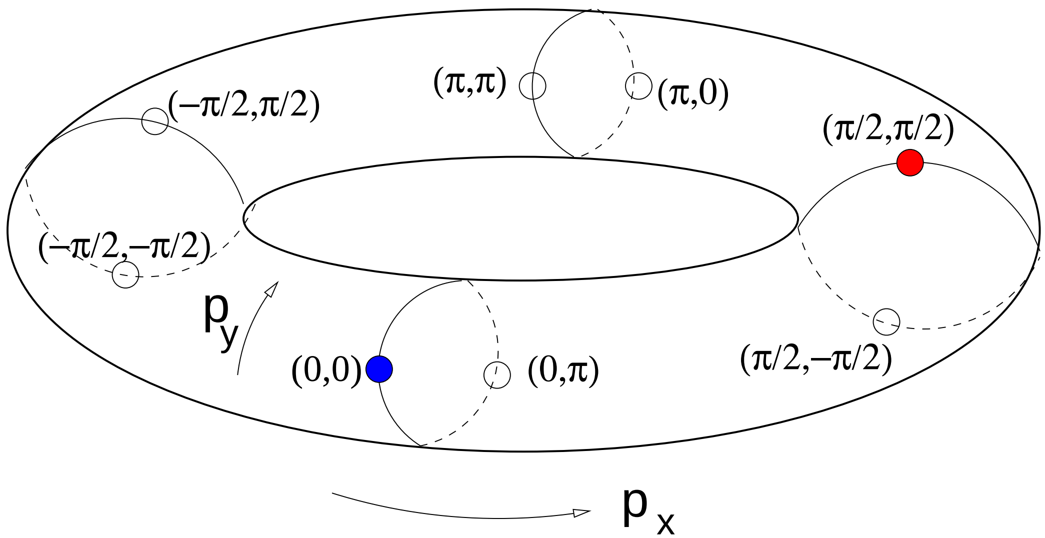
Complimentarity: $D(p_\mu = \pi/2) = \overline{D}(p_\mu = 0) = 4i\Gamma$

Combine the naive actions

$$\boxed{\mathcal{D} = D + \overline{D} - 4i\Gamma}$$

Free theory

- $\mathcal{D}(p) = 2i\sum_\mu \left(\gamma_\mu \sin(p_\mu) + \gamma'_\mu \sin(\pi/2 - p_\mu)\right) - 4i\Gamma$
- at $p_\mu \sim 0$ the $4i\Gamma$ term cancels $\overline{D}$, leaving $\mathcal{D}(p) \sim \gamma_\mu p_\mu$
- at $p_\mu \sim \pi/2$ the $4i\Gamma$ term cancels D, leaving $\mathcal{D}(\pi/2 - p) \sim \gamma'_\mu p_\mu$
  - Only these two zeros of $\mathcal{D}(p)$ remain!



THEOREM: these are the only zeros of $\mathcal{D}(p)$ (appendix)

- at other zeros of D, $\overline{D} - 4i\Gamma$ is large
- at other zeros of $\overline{D}$, $D - 4i\Gamma$ is large

Chiral symmetry remains exact

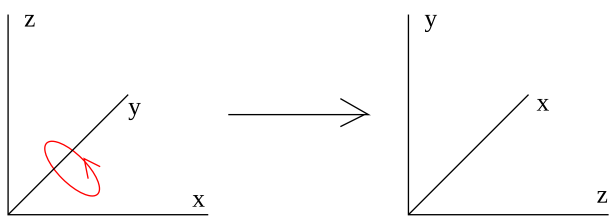
- $\gamma_5 \mathcal{D} = -\mathcal{D}\gamma_5$
- $e^{i\theta\gamma_5}\mathcal{D}e^{i\theta\gamma_5} = \mathcal{D}$

But

- $\gamma_5' = \Gamma\gamma_5\Gamma = -\gamma_5$
- two species rotate oppositely
- symmetry is flavor non-singlet

Space time symmetries

- usual discrete translation symmetry
- $\Gamma = \frac{1}{2}\sum_\mu \gamma_\mu$ treats primary hypercube diagonal specially
- action symmetric under subgroup of the hypercubic group
  - leaving this diagonal invariant
- includes Z_3 rotations amongst any three positive directions
  - $V = \exp((i\pi/3)(\sigma_{12} + \sigma_{23} + \sigma_{31})/\sqrt{3})$ $\qquad [\gamma_\mu, \gamma_\nu]_+ = 2i\sigma_{\mu\nu}$
  - cyclicly permutes x_1, x_2, x_3 axes $\qquad [V, \Gamma] = 0$
  - physical rotation by $2\pi/3$



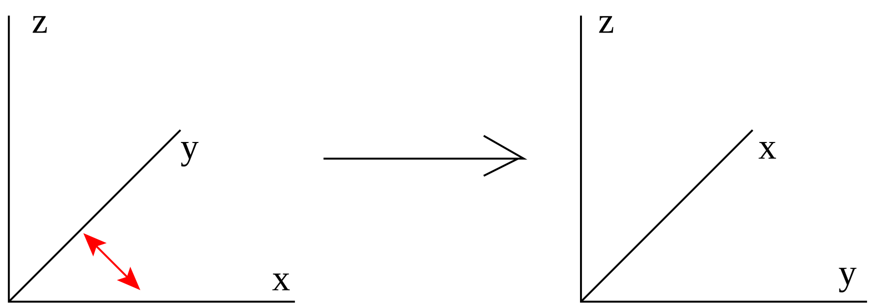
  - $V^3 = -1$: we are dealing with fermions

Repeating with other axes generates the 12 element tetrahedral group

- subgroup of the full hypercubic group

Odd-parity transformations double the symmetry group to 24 elements

- $V = \frac{1}{2\sqrt{2}}(1 + i\sigma_{15})(1 + i\sigma_{21})(1 + i\sigma_{52})$ $\qquad [V, \Gamma] = 0$
- permutes x_1, x_2 axes
- $\gamma_5 \to V^\dagger \gamma_5 V = -\gamma_5$



Natural time axis along main diagonal $e_1 + e_2 + e_3 + e_4$

- T exchanges the Fermi points
- increases symmetry group to 48 elements

Charge conjugation: equivalent to particle hole symmetry

- $\mathcal{D}$ and $\mathcal{H} = \gamma_5 \mathcal{D}$ have eigenvalues in opposite sign pairs

Special treatment of main diagonal

- interactions can induce lattice distortions along this direction
  - $\frac{1}{a}(\cos(ap) - 1)\overline{\psi}\Gamma\psi = O(a)$
  - symmetry restored in continuum limit
- at finite lattice spacing can tune — Bedaque Buchoff Tiburzi Walker-Loud
  - coefficient of $i\overline{\psi}\Gamma\psi$ — dimension 3 operator
  - 6 link plaquettes orthogonal to this diagonal
- zeros topologically robust under such distortions
  - Nielsen Ninomiya, MC

Appendix A: Proof that there are only two zeros of $\mathcal{D}(p)$

- $\mathrm{Tr}\ (\gamma_\mu - \gamma_\nu)\mathcal{D}(p) \sim \sin(p_\mu - \pi/4) - \sin(p_\nu - \pi/4)$
  - at a zero: $\cos(p_\mu - \pi/4) = \pm\cos(p_\nu - \pi/4)$
  - all cosines equal in magnitude
- $\mathrm{Tr}\ \Gamma\mathcal{D}(p) = 0\ \Rightarrow\ \sum_\mu \cos(p_\mu - \pi/4) = 2\sqrt{2} > 2$
  - all cosines positive
  - at a zero: $\cos(p_\mu - \pi/4) = +1/\sqrt{2}$

All components of p_μ are equal and either 0 or $\pi/2$

Appendix B: Actions from Karsten and Wilczek

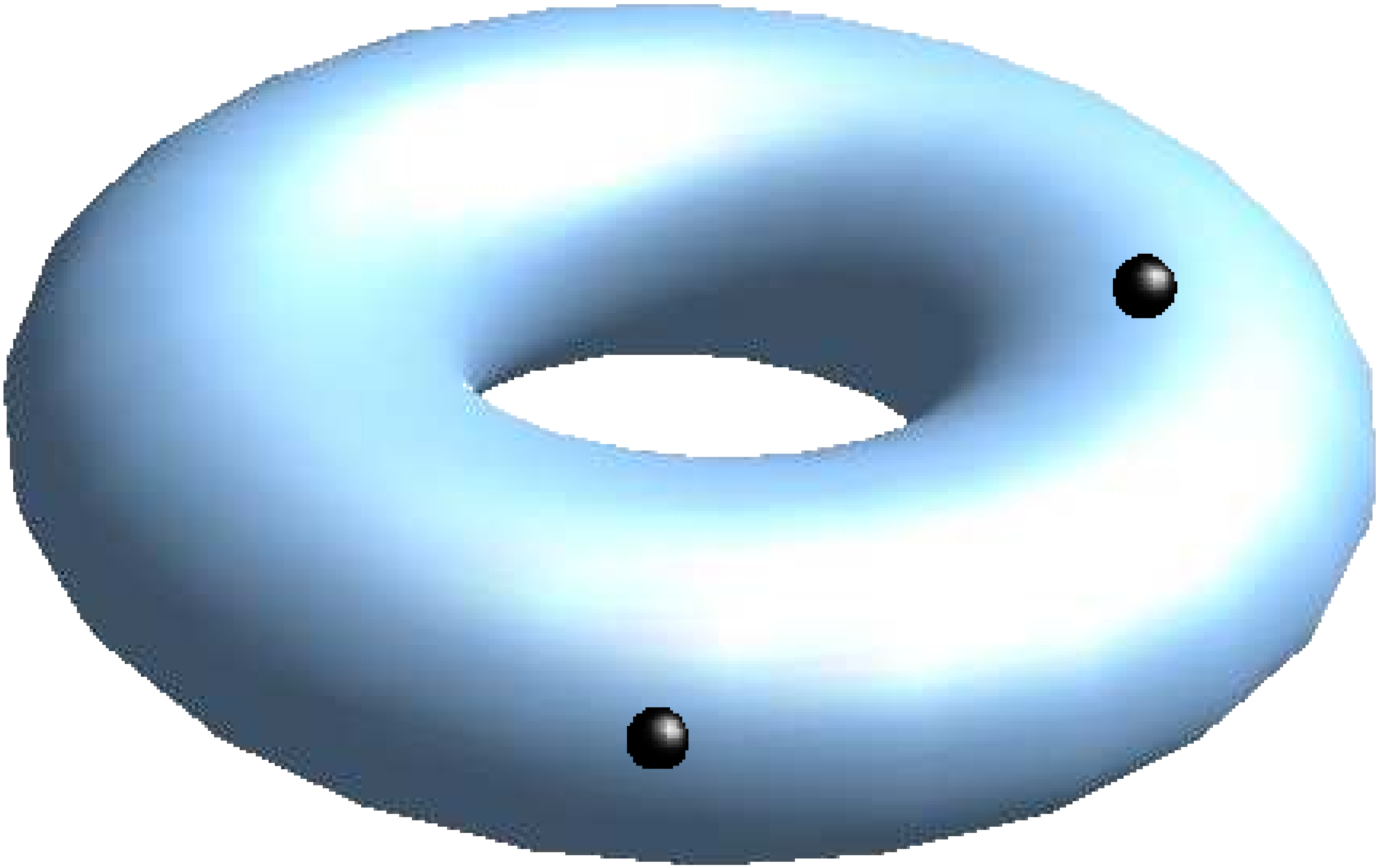
- both equivalent up to a unitary transformation
  - $\psi \to i^{x_4}\psi$

$$D = \sum_{\mu=1}^{4} \gamma_\mu \sin(p_\mu) + \gamma_4 \sum_{i=1}^{3} (1 - \cos(p_i))$$

- last term removes all zeros except $\vec{p} = 0$, $p_4 = 0, \pi$

Now x_4 chosen as the special direction

- onsite term $\sim \gamma_4$ instead of $\sim \Gamma$
- $\vec{\gamma}' = \vec{\gamma}$, $\gamma_4' = -\gamma_4$
  - $\gamma_5' = -\gamma_5$



"Here’s what I’ve learned: that you can’t make fun of everybody, because some people don’t deserve it." -- Don Imus